TRANSPORTATION MODELING FOR REGIONAL PLANNERS

ASSESSMENT OF THE DEMAND FOR ROUTES AND TRAFFIC STREAM ASSIGNMENT ON INTER-REGIONAL TRANSPORTATION NETWORKS (♦)

Summary [*]

It is a mathematical simulation model, which aims at providing transportation and regional planners with a probabilistic method for the theoretical assignment of traffic flows to routes of any inter-regional transportation system.

The system referred to by the model may consist of various transportation infrastructure and modes, which connect settlements in large geographical regions.

The basic criteria used to construct the model respond to the need for

(1) a minimisation of the input data set;

(2) a probabilistic (instead of deterministic) approach to the assessment of the users' demand for alternative travel routes and modes;

(3) simulating cases of possible traffic congestion in any route section, along with the simulation of the relevant "spontaneous" solution.

The probabilistic approach exploits the amount of *uncertainty* that is always associated with any probability distribution. This criterion avoids resorting to the optimisation of given "objective-functions", as is instead usual to the majority of the other traffic assignment models.

This assignment model is also viewed as an instrument fit for deepening the analysis of an important feature of regional systems, considering that traffic streams are a major physiological aspect of any set of inter-related human settlements.

Configuration and evolution of transportation infrastructure and traffic streams are crucial study subjects for any regional analysis and development planning.

Introduction

The model makes a clear distinction between the assessment of the users' probable demand for travel routes and the actual distribution of traffic streams that is reasonable to expect.

The distinction is necessary to understand and manage possible incompatibility between the travel conditions expected by the system's users and the physical/functional constraints posed by the available transportation system.

"Demand for route/mode" means the amount of traffic units that opt - within a given time unit - for a particular route/mode, among a range of alternative options which make it possible to reach the same destination starting from a given travel origin.

Any user of the transportation system chooses the routes and the modes that are subjectively considered as the most convenient ones, according to an evaluation of convenience that is often based on incomplete or inadequate information. This is one of the probable causes of traffic congestion, though it is appropriate assuming that part of the system's users are ready to take the risk and the cost of a certain degree of traffic congestion in some sections of the chosen routes.

The model intends to simulate

(i) the formation of the user's choice as to the route to go, considering the basic information the user may possess;

(ii) the expectable consequence of the users' demands for routes on the actual distribution of traffic streams among the routes of the network.

The *user* that is thought of by the model is obviously an abstract *average user*, who undertakes any travel in view of a benefit – attained at the travel's destination – which is compared to the expected cost of the travel to undertake.

Alternative routes and modes provide the user with a prospect of balances between benefits and costs, i.e., with a set of possible options associated with a range of respective degrees of satisfaction expected by the user.

It is reasonable to assume that the probability of the user's choice is related to the degree of satisfaction expected from each possible option.

Another assumption is that the transportation system's users may only exceptionally modify their choice during the transfer. Modifications of travel routes during the transfer may mostly depend on unforeseen levels of traffic congestion. The congestion occurs when the confluence of the users' demand for the route exceeds the route's physical capacity.

An important feature of the congestion analysis is the critical difference accounted for between the concept of *route saturation regime* and the concept of *route congestion regime.*

The traffic congestion analysis carried out by the simulation model re-assigns users rejected by too high traffic congestion to alternative options, according to the respective degree of probability.

The congestion analysis is based on traffic flow equations and diagrams, which - for each type of infrastructure and transportation mode - put in mutual relationship traffic speed, traffic density, capacity and other characteristics of the route, stream intensity, congestion levels, and tolerable degree of congestion.

(♦) This essay expounds an independent revision and theoretical development of a partly similar transportation model that was outlined in 2003 in a cooperation with **Gianfranco Antonini**, transportation expert to whom I owe a major debt of gratitude. The new contents of this essay are based on procedure adjustments, results and suggestions that come from a very long series of laboratory and field tests made by G. Antonini on the 2003 version of the model.

[*] *The List of Contents is at Page 26 (last page).*

Transported loads belong to different categories, such as – only to mention the two principal ones – passengers and goods.

The traffic assignment may require that streams consisting of various load categories be separately considered on each route of the transportation network. In that case, the model's input data must be diversified by load category, though, for computational purposes, the measurement of different categories of load must be made homogeneous through the adoption of appropriate conventional measurement units.

The logical structure and the characteristics of the model allow its operator to summarise the results of the model application in an ample range of significant indicators, which can facilitate decision making in both transport and regional planning activities.

1 - DEFINITIONS AND TERMINOLOGY

1.1 – The Transportation Network

In addressing the study subject with abstract theoretical instruments, the infrastructure system of any interregional transportation network may schematically be represented by a graph.

A graph is a geometrical figure consisting of a set of scattered dots connected with each other by linear segments named "arcs".

In the model, the *arcs represent sections of the routes* that connect any travel origin to any travel destination.

The meaning of the dots of the graph changes according to what they represent. In fact, the dots may represent two different functions of two different components of any transportation network. The dots of the graph that represent origins and/or destinations of traffic streams are marked as *poles*. The dots of the graph that indicate only junctions of *arcs* are marked as *nodes*.

It is worth observing that every *pole* is also a node, since it is always either at a junction of arcs or at an arc-end**;** whereas a simple *node*, which is a mere junction of arcs, is never meant to be a pole.

Each pair of nodes may be connected by more than one type of infrastructure and, therefore, by more than one arc.

Moreover, and in the model representation, each arc is an oriented line, which represents a canal that conveys only a one-way traffic flow. The line orientation indicates the traffic stream direction allowed by the canal. For example, a two-way road between two nodes N_1 and N_2 is represented by two different arcs of the graph, one for the "canal" that conveys the traffic stream from N_1 to N_2 , the other to represent the "canal" that conveys the stream from N_2 to N_1.

1.2 – Routes and Links – Transport and Travel

1.2.1 - In this context, the term "route" means a chain of *compatibly oriented* arcs of infrastructure that connect *two poles*. The chain of arcs is *compatibly oriented* if each arc has the same direction of the traffic stream conveyed.

Any *route* may consist of more than one type of infrastructure (for example, may consist of a road, or of a railway, or of a chain of road arcs combined with railway arcs, etc.)

"Demand for route" means the amount of traffic units that opt - within a given time unit - for a particular route, among a range of alternative options which make it possible to reach the same destination starting from a given travel origin.

1.2.2 - The term "link" is here used to mean a *given set of routes* that connect any *oriented pair* of poles. Given two poles, *A* and *B*, the oriented pair *A_B* is different from the oriented pair *B_A*, since the set of routes from *A* to *B* may in general be different from the set of routes from *B* to *A*.. This necessarily implies that "link" *A_B* differs conceptually from "link" *B_A* . Therefore, the concept of "link" is substantially coincident with the concept of "oriented pairs of poles".

1.2.3 – The model uses the word "travel" to mean the journey of anything along one of the routes that connect any travel origin pole to any travel destination pole. Therefore, any *travel* identifies also a particular route of the relevant link.

1.2.4 - Since the word "transport" is normally used to mean a displacement of people or goods from one place to another irrespective of the route gone, the same term is here used to mean *the possibility* of travel allowed by the set of routes of any given link.

In this context, the term "transport" may be used in expressions like "*demand for transport"* or *"transport demand"*, to denote the demand for the use of any transportation link. For instance, the *demand for transport on L*, or – more simply – *the demand for L*, means the overall transport demand by use of link *L*.

Thus, "demand for transport" or "transport demand" means the amount of traffic units bound to engage a given link in a given unit of time. Concerning the whole transportation system, the expression "overall demand for transport" is used to mean the sum of the transport demands that engage all the links of the study system.

1.2.5 - The expression "traffic flow" or "traffic stream" means the *number of travels*, or of *traffic units,* that pass through a hypothetical cross section of any arc of route in a specified time unit.

The expression "traffic flow" is however a little more general than "traffic stream", for it may also regard the traffic flow in a link, i.e., the total transport volume moving in all the link's routes in the given unit of time.

"Transport or traffic units" are conventional units used to homogenize the measurement of any amount of transported load irrespective of its physical nature, according to technical criteria proper to transportation engineering. The most common transport unit used in transportation studies is the *passenger car unit* ("*pcu*").

Nodes, poles, arcs, routes and links are the main fixed components of any transportation network. Each component is characterized by a specific set of parameters that form the basic set of the model's input.

If the network includes N poles, then $N(N\text{-}1)$ is the maximum number of possible transports between different poles of the same network (in a given time unit).

The overall number of possible streams depends instead on the number of routes pertaining to the links.

For the sake of simplicity, transports and travels internal to each pole (for example, from A to A itself) are not allowed for in this text. (Such an omission doesn't affect at all the logic of the model expounded here. Internal transports and travels could be introduced whenever required, though paying a special attention to the definition and significance - within urban areas - of internal poles, internal routes and internal nodes).

1.3 – Transport Demand and Actual Streams

The model aims at formulating and solving the following problems:

(a) – Given the *unit mean transportation cost* relevant to every route of the system, what – in a fixed time unit – is the *probable traffic demand for each route*?

(b) – Once assessed the probable *demand* for each route of the network, what – in the given time unit – is the *actual volume of traffic stream* to be expected on the same route?

Problem (a) above is different from problem (b). The reason is that the most demanded routes are also the routes where traffic tends to concentrate. The degree of traffic concentration in a particular route may bring the *traffic flow,* or *stream* (which is expressed in terms of transportation units moving per time unit toward their destination) to drop to a level lower (or much lower) than the maximum allowed by the physical capacity of the route.

In other words, this means that the most demanded routes *do not necessarily* accommodate the greatest *volumes* of traffic.

To grasp the sense of this statement, it is sufficient to imagine a road route congested to the point that no vehicle can move towards its destination just because of an excessively high concentration of vehicles there. In a situation like that, the traffic volume, in terms of traffic flow, is nil.

Moreover, one must consider that the maximum traffic flow allowed by the physical capacity of any given route does not necessarily correspond to the optimum flow expected by the users.

The maximum flow, in fact, is normally not associated with the optimum trip speed/duration planned by the users. Travel times tend to increase with increasing volumes of traffic.

The fear of encountering a traffic flow close to its maximum may induce users to choose, instead of the theoretically best route, "*inferior*" alternative routes, which can nevertheless allow shorter travel times.

For these reasons, the model systematically distinguishes *demand for route* from *actual stream expected on the same route.*

In any case, it must be borne in mind that the model aims only at assessing the *probability* of both traffic demand and actual traffic stream, in consideration of the non-deterministic approach adopted.

2 – THE STRUCTURE OF THE MODEL

2.1 – Functional Characteristics

The model can be used to simulate the probable distribution of *aggregate streams* of traffic and of *disaggregate streams* of traffic.

Aggregate streams do not account for the various kinds of transported loads. The *aggregate model* aims only at assessing the likely distribution of the traffic load among the system's routes, without specifying the nature of each volume of traffic in terms of different amounts of different transported things.

Instead, *disaggregate streams* account for the various kinds of transported loads.

If the main purpose is to assess *disaggregate streams*, the model must develop according to two subsequent phases**:** in the first phase the model assesses aggregate streams of traffic, which become input of the *disaggregate model* that leads – in the second phase – to the assessment of *disaggregate* streams.

The second phase of the model may itself be considered as a model of *disaggregate distribution* of traffic streams, though it should not be used but as a logical extension of the aggregate model.

However, under a higher number of additional constraints and with a lower logical consistency, the *disaggregate model* can be used as an autonomous model, i.e., without passing through the aggregate model. This particular use of the *disaggregate* model will be discussed later.

The possibility of using the model for both aggregate and disaggregate assignments depends only on the availability of the data that concern the mean unit transportation costs relevant to each kind of load transported on each route of the network.

2.2 – Basic Assumptions

The model construction rests on the following fundamental hypotheses**:**

(1) as to any possible travel destination, the users of the transportation system consider only a limited and clearly identified selection of transportation routes and modes. Therefore, before starting any application of the model, it is necessary to identify the possible routes and modes relevant to each link;

(2) the system's users are aware both of the possible advantages and of the basic costs associated with each of the possible travel options;

(3) every user knows that any route minimum cost (or – from now on – "**free-route cost**" or "**technical cost**") is systematically increased by the volumes of traffic expected on the route. The model schematizes this hypothesis by the following definition of "mean *conditioned* unit costs":

[a] $J_i^L = C_i^L\,[1 + k\mathbf{Ln}(d_i^L / \underline{d}\,)],\qquad \forall\ i, L,$

in which

J_i^L indicates the mean ***conditioned*** cost per transported unit along route *i* of link *L;*

C_i^L indicates the mean *technical cost* (or *free-route* cost) per transported unit along the same route;

d_i^L represents the volume of the traffic stream on the route as expected by the user;

$\underline{d} = 1$ is the appropriate traffic measurement unit, chosen according to the characteristics of the study system**;**[1]

$k > 0$ is a constant multiplier proper to the study system;

Ln stands for "natural logarithm".

In the ***disaggregate*** model, hypothesis **[a]** becomes **[a₁]** by substitution of the symbols J_i^L and C_i^L with the two symbols $J_{\alpha\,i}^{\ L}$ and $c_{\alpha\,i}^{\ L}$, respectively, which represent the corresponding *conditioned* and *technical costs* per unit transported load of **category *α*** on route *i* of link *L,* whatever category *α* .

It's worth stressing that in the disaggregate model the values to consider for both ratio $d_i^L / \underline{d}$ and coefficient *k* are the same ones as for the aggregate model.

2.3 – Basic Input Data

The basic data necessary to use the model are taken from the statistics of facts and events relevant to the study subject.

Amongst the data, of a primary importance are**:**

(a) The accurate identification of the set of the viable alternative routes of each link;

(b) A set of figures relevant to all the system's links, which – for any link *L* – includes **either** the *transport demand* F^L (i.e., the total transport demand that engages link *L*) **or** the *travel demand* for one (anyone) of the link's routes (remembering that link A-*to-B* ≠ B-*to*-A, whatever *A* and *B*));

(c) The transportation costs per transported unit, which are associated with each route of the system, in relation to the physical standards of the route, to the characteristics of the transportation vehicles, and to the measurement system adopted for the traffic streams;

(d) The values obtained through field surveys for the traffic *streams* relevant to *any two* routes of ***only one*** (anyone) of the system's links.

The data concerning these "given" *streams* regard *aggregate flows* (i.e., whatever the vehicles and loads may be) in the *aggregate model*; whereas, in the *disaggregate model,* the data regard the *disaggregate streams* of the different categories of vehicles and transported loads on the two routes, surveyed independently of each other.

The link and the routes regarded by the survey, together with the relevant two traffic streams surveyed, are referred to as "*reference link*", "*reference routes"* and *"reference streams"*, respectively.

The need for these *two* particular data, instead of *one single datum* as per (b) above concerning all the other links, is an exception that regards the "reference link" only.

2.4 – Streams and Probable Demand for Route

The *total traffic flow* that, in any given time unit, moves from one pole to another consists of the sum of all the *streams* that engage the whole set of selected alternative routes connecting the two poles.

It's worth recalling that the term "stream" indicates the amount of traffic engaging *one single route* in the time unit.

[1] The number of travels to account for in any large transportation system is normally very high. The travel stream measurement units may vary also according to the time units considered. This is why, using for instance units such as *pcu/hour* or *pcu/day* or *pcu/week*, the appropriate unit value for $\underline{d}$ could be either hundred or thousand or more *pcu* per time unit.

Because of the preceding definitions, it is possible to write the following equivalences:

[1] $\Sigma_i f_i^L = F^L$, $\{i = 1, \dots r^L;\ L = 1, \dots, N(N-1)\}$.

in which f_i^L is the stream in any route i of the r^L routes of link L;

[2] $\Sigma_L \Sigma_i f_i^L = \Sigma_L F^L = F$

in which F is the overall traffic flow engaging the whole system's network in the time unit considered.

The general problem that the model intends to solve is the determination of the distribution of all the streams among the whole set of routes of the system's network, which means the determination of the expected value for every f_i^L.

The solution to the problem is provided by two subsequent steps.

The first step accounts for the data incompleteness. The available information is supposed to be the more or less accurate information shared by most network users, so that each "datum" must be taken as a coarse mean estimate of many different subjective estimates.

On this basis, and ***given*** *the set of data* indicated in the preceding paragraph, it is only possible to make a tentative probabilistic estimate of the *demand for route* relevant to each route of the system.

Such a *probable demand for route* is only a preliminary/subjective theoretical assessment based on the maximum amount of information that users may normally have at their disposal. As already mentioned in the preceding paragraph, this demand for route does *not necessarily* correspond to the *actual* stream to be expected in each route. The actual intensity of the stream depends also on the route capacity, which is not always in condition to accommodate all the relevant traffic flow demanded.

Then, in the second step of the problem solution process, each *given transport demand* is compared with the respective set of both estimated demand for routes and the capacities of these, to find a logical criterion for assessing the most likely traffic flow distribution to be expected in each link.

In this connection, it is important to observe that different routes of different links may have arcs in common.

Taken for granted the conceptual distinction between *demand for route* and *actual stream in the same route*, it is nevertheless worth remarking that in one particular case only – i.e., only concerning the two given streams measured by direct survey – *demand for route* and *stream* do necessarily coincide.

2.5 – Mean Transport Unit Cost

The transportation mean unit *basic* cost relevant to any particular route is supposed to be more or less effective in traffic conditions that are far from the route saturation. This mean unit cost, as already pointed out, is referred to by expressions such as "free-route cost" or "technical cost".

The analysis of the network characteristics, along with the use of statistical data and standard engineering parameters, leads to the identification of such free-route (or "technical") mean unit costs.

Symbol C_i^L is here used to denote the *free-route mean transportation unit cost* for using route i of link L.

Users know this cost may only exceptionally be the actual one. Any network user does normally expect that the actual transportation cost (at least in terms of time) is also conditioned by the expected intensity of the traffic on the route.

It is therefore reasonable to hypothesize that the expected average transportation unit cost on route i of link L is also a function of the overall demand for the same route. As anticipated by hypothesis (3)**[a]**, Paragraph **2.2,** the model formalizes the resulting cost function as follows:

[3] $J_i^L = C_i^L (1 + k\, \mathbf{Ln} d_i^L)$, $\forall\ i, L,$

in which k is a positive constant proper to the study system, **Ln** is the operational symbol for "natural logarithm", and d_i^L is the expected demand for route i of link L.

Coefficients « $(1 + k\, \mathbf{Ln} d_i^L)$ » are referred to as "*conditioning factors",* while mean unit costs J_i^L are referred to as "conditioned costs".

Definition J_i^T is a working hypothesis, which uses logarithms upon the assumption that no demand for route is less than one (i.e., no route of L is engaged by less than one transportation unit per time unit: logarithms of quantities less than 1 are negative numbers, which would make no sense for conditioning factors).

By the way, this might also be the criterion for identifying the set of eligible alternative routes of any link.

Thus, if J_i^L is the expected actual mean unit cost, the cost to be associated with the whole demand d_i^L is expressed by:

[4] $\Gamma_i^L = J_i^L d_i^L = C_i^L (1 + k\, \mathbf{Ln} d_i^L)\, d_i^L$, $\forall\ i, L.$

Demands d_i^L , whatever i and L, represent the first set of the problem's unknowns.

2.6 –Travel Benefit

According to hypothesis (2) introduced in Paragraph **2.2** above, any user of the transportation system is expecting a net benefit from the travel to make.

If B^L is the expected mean *gross* benefit associated with any *unit* of transportation demand for link L, then the total *net* benefit U^L to be associated with the overall transport demand F^L on link L can be expressed by

[5] $U^L = B^L F^L - \boldsymbol{\Sigma_i}\, \Gamma_i^L = \boldsymbol{\Sigma_i}[B^L - C_i^L(1 + k\, \mathbf{Ln} d_i^L)]\, d_i^L$,

on the basis of Equations **[1]** and **[4]** and of the consideration that, by definition, is

[6] $\boldsymbol{\Sigma_i}\, d_i^L = \boldsymbol{\Sigma_i} f_i^L = F^L$.

i.e., in every link, the sum of route demands (the *transport demand* on the link) coincides with the sum of the travel streams.

A definition of the total net benefit U associated – per time unit – with the network activity, can be derived from **[5]** above, according to the following equation

[7] $U = \boldsymbol{\Sigma_L}\, U^L = \boldsymbol{\Sigma_L \Sigma_i}[B^L - C_i^L (1 + k\, \mathbf{Ln} d_i^L)]\, d_i^L$.

For model building purposes, quantities B^L and U are *shadow data*. Such quantities, in the mathematical process that follows, are either eliminated or incorporated into other compound quantities that can easily be determined.

2.7 – A Probabilistic Approach

As previously remarked, the first problem is to determine the *probable* distribution of route demand d_i^L among all the routes of the system.

The option for the probabilistic approach requires that the unknown quantities d_i^L be translated into *probabilities of demand.*

For this purpose, and from a merely formal point of view, instead of dealing with d_i^L directly, it is sufficient to deal with the ratios expressed by

[8] $p_i^L = d_i^L / F$, $\{i = 1,\ldots r^L\,;\ L = 1,\ldots, N(N-1)\}$.

Each relation of these defines variable p_i^L as the *probability – in the fixed time unit – that any unit of route demand in the system is for route* i *of link L.*

Definition **[8]** verifies the following condition:

[9] $\boldsymbol{\Sigma_L \Sigma_i}\, p_i^L = \boldsymbol{\Sigma_L \Sigma_i}\, (d_i^L / F) = \boldsymbol{\Sigma_L \Sigma_i}\, (f_i^L / F) = 1$ **;**

this allows us to consider all probabilities p_i^L as elements of a discrete probability distribution.

Condition **[9]** facilitates the solution to the problem, considering that the number of the unknowns is much greater that the number of equations involving the same unknowns.

As shown later, the determination of p_i^L implies the determination of d_i^L, by use of definition **[8],** since

[10] $p_i^L = d_i^L/F$, whence $d_i^L = F\, p_i^L$,

F being the sum of the whole set of route demands in the system (i.e., the overall traffic demand in the given time unit).

Among the properties of probability distributions, there is one of a particular importance for this model.

According to a well-known theorem, proved by Shannon and Weaver in 1949, a specific quantity, labeled as *uncertainty E*, is always associated with any *probability distribution*. In this context, such a quantity can be expressed by

[11] $E = \mathbf{-}\, \boldsymbol{\Sigma_L \Sigma_i}\, (p_i^L\, \mathbf{Ln}\, p_i^L)$.

Uncertainty E is either a positive or a nil quantity. Sign "-", which affects the right hand side of definition **[11],** is to turn $\mathbf{Ln}\, p_i^L$ into positive values, for p_i^L, in any probability distribution, are less than or equal to 1.

If $\boldsymbol{\Pi}$ is the number of different probabilities of which the distribution consists, then the absolute maximum uncertainty is expressed by

[12] $E_{max} = \mathbf{Ln}\, \boldsymbol{\Pi}$

which is given by a probability distribution where all of the $\boldsymbol{\Pi}$ probabilities have identical value $1/\boldsymbol{\Pi}$.

Uncertainty E is nil when one only expected event is possible, which is therefore a certain event.[2]

Any information that is available about probability distribution p_i^L reduces the amount of associated uncertainty, to the extent that the available information can be expressed by equations that involve some or all of the distribution probabilities.

As a consequence of this, the determination of the probabilities can be obtained by calculation of the maximum amount of uncertainty that remains associated with the probability distribution under a set of given constraints. These constraints are formed by the equations that bind some or all of the distribution probabilities to verify certain given conditions.

It must be observed that, by hypothesis, all the available information is expressed by the constraint-equations, which means that the uncertainty is objecti-

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[2] Uncertainty (or *entropy*) E is also nil when no event is possible and, therefore, all probabilities are nil.
In fact, for any probability p, it is : $\lim_{p \to 0} (p\, \mathbf{Ln}\, p) = 0$.

vely *maximum* as for the rest. No information can be utilized by the model if it cannot be translated into one or more constraint equations.

This logical approach to the problem is fully justifiable only if the number of mutually independent constraint-equations is less than the number of the unknown probabilities to be determined.

3 – THE AGGREGATE MODEL

3.1 – Assigning Travel Demand

Summarizing the preceding observations, the determination of stream probabilities p_i^L becomes the problem of determining the maximum value for E, i.e., the maximum value of function **[11]** under the constraints provided by Equations **[6]** and **[7],** after dividing these by F, to turn demands d_i^L and F^L into probabilities $p_i^L = d_i^L/F$, and $P^L = F^L/F$, respectively.

The determination of the *constrained maximum* of any mathematical function can be made according to various methods, of which the *Lagrange Multipliers Method* is the most usual.

By adoption of this method, referring to Equations **[6]** and **[7]** after considering definition **[10]**, the problem to solve becomes:

Determine

[13] $$E = -\boldsymbol{\Sigma_L \Sigma_i} (p_i^L \,\mathbf{Ln}\, p_i^L) = \text{maximum}$$

under the following $1+N(N-1)$ constraint equations:

[14] $$Y^L = \boldsymbol{\Sigma_i} p_i^L - P^L = 0, \qquad \forall L,$$

[15] $$Z = \boldsymbol{\Sigma_L \Sigma_i}[B^L - C_i^L(1 + k\mathbf{Ln}F + k\mathbf{Ln}p_i^L)]p_i^L - \underline{U} = 0.$$

3.1.1 – Theorem 1:

< The solution to the problem is expressed by

[16] $$p_i^L = A\,\mathbf{exp}(Q^L / x_i^L), \qquad \forall i, L,$$

in which

[17] $$A = 1 / F\,\mathbf{exp}(1+1/k),$$

is a constant inherent in the study system.

If two aggregate reference streams, f_m^R and f_n^R, relative to any couple of reference routes m and n of any reference link R, are known [according to the requirement of Paragraph **2.1-**(d)], then system constant multiplier k can be determined by the solution of the following second degree equation:

[18] $$\alpha k^2 + \beta k + \gamma = 0$$

in which

[19]

$$\alpha = C_m^R C_n^R/(C_m^R - C_n^R) - C_m^R\,\mathbf{Ln}ef_{\boldsymbol{m}}^R +$$
$$- \{[(C_m^R + C_n^R)\,\mathbf{Ln}(f_m^R/f_n^R)/(C_m^R - C_n^R)] +$$
$$- \mathbf{Ln}ef_1^R\}(C_n^R\,\mathbf{Ln}ef_n^R - C_m^R\,\mathbf{Ln}ef_m^R) +$$
$$- (C_n^R\,\mathbf{Ln}ef_{\boldsymbol{n}}^R - C_m^R\,\mathbf{Ln}ef_m^R)^2/(C_m^R - C_n^R);$$

$$\beta = 3(C_n^R\,\mathbf{Ln}ef_n^R - C_m^R\,\mathbf{Ln}ef_m^R) + C_1^R\,\mathbf{Ln}(f_m^R/f_n^R);$$

$$\gamma = (C_n^R - C_m^R),$$

$\boldsymbol{e} = 2.7182818\ldots$ being the base of natural logarithms.

Normally, the solution of Equation **[18]** provides two values for multiplier k, of which only the positive solution is the significant one.

Concerning any link L (obviously including R), it must be pointed out that

[20] $$x_i^L = 1 + u\,C_i^L, \qquad \forall i, L,$$

[21] $$Q^L = \mathbf{Ln}(p_m^L / p_n^L)/(1/\boldsymbol{x}_m^L - 1/x_n^L),$$

for any m, n, L.

Q^L is a constant characteristic of each "link" (to be calculated through a series of polynomial equations for any link $L \neq R$), whereas

[22] $$u = \mathbf{Ln}(p_m^L / p_n^L) / [C_n^L\,\mathbf{Ln}(p_n^L/A) - C_m^L\,\mathbf{Ln}(p_m^L/A)]$$

is a system constant, for any m, n and L. >

Proof: The proof of the above theorem consists of the application of the method of Lagrange multipliers to the maximization of **[13]** under constraints **[14]** and **[15].**

3.1.2 - As to the "reference link R", Equations **[21]** and **[22]** become

[21a] $$Q^R = \mathbf{Ln}(f_m^R / f_n^R)/(1/x_m^R - 1/x_n^R), \quad \text{and}$$

[22a] $$u = \frac{\mathbf{Ln}(f_m^R / f_n^R)}{C_n^R\,\mathbf{Ln}f_n^R - C_m^R\,\mathbf{Ln}f_m^R + (C_n^R - C_m^R)(1+1/k)},$$

respectively, after application of definitions **[8], [17]** and Equation **[16],** f_m^R and f_n^R being the two streams determined by direct field survey.

3.1.3 – Theorem 2:

< Given all the mean "technical" costs, the system constants k and u, and any *one* route demand for any link of the system that is different from the "reference link", all the other route demands inherent in the same link remain determined along with the total transport demand for the link >.

Proof: Remembering definitions **[8]** and **[17]** for p_i^L and A, respectively, from Equation **[22]** we obtain

[23] $$\mathbf{Ln}\,d_n^L = \frac{(1+uC_m^L)\mathbf{Ln}\,d_m^L - u(C_n^L - C_m^L)(1+1/k)}{(1 + uC_n^L)}$$

whatever L, m, n.

If d_m^L is the *given* route demand, then all the other route demands d_n^L can be calculated by use of **[23],** after passing from the logarithms to the respective numbers.

Subsequently, also transport demand F^L on the link can obviously be calculated by

$$F^L = \Sigma_n d_n^L, \qquad (n = 1, 2, \ldots, r).$$

Therefore, *Theorem 2* is demonstrated, irrespective of *r,* i.e., of how many routes, and streams, belong to the same link.

***3.1.4* -** It remains clear that the calculation of all the route demands relevant to any link is also possible by use of Equations **[16],** if transport demand F^L is the known datum for the link [refer to Paragraph **2.3** (b)]. In this case also link constants Q^L must be calculated before applying Equations **[16].** It is also clear that Equations **[21]** cannot be used for the purpose, because – in such cases – no route demand is given. Therefore, it is necessary to resort to the set of polynomial equations that can be derived from Equations **[16],** considering that

$$\Sigma_i p_i^L = F^L/F = \Sigma_i \exp(Q^L / x_i^L)/ F \exp(1+1/k), \quad \text{i.e.,}$$

[16a] $$F^L \exp(1+1/k) = \Sigma_i \exp(Q^L / x_i^L), \qquad \forall L.$$

If, in the above equations, one poses

$\exp(Q^L) = z^L$, and $1/x_i^L = y(iL)$, it is possible to re-write Equations **[16a]** in the following form:

[24] $$\Sigma_i (z^L)^{y(iL)} = F^L \exp(1+1/k), \qquad \forall L,$$

in which exponents $y(iL)$ are all known quantities, and z^L is, in each of the Equations **[24]**, the unknown to be determined.

The solution to these equations is easily possible through iterative procedures. Once each z^L has been calculated, the searched respective value is $Q^L = \mathbf{Ln}\, z^L$.

3.2 – Transport Demand for the Reference Link

Theorem 2 is not valid for the "reference link", because this theorem is applicable only after the determination of network constants *k* and *u.* The two surveyed streams of the reference link are necessary just for this purpose.

After the determination of system constants *k* and *u,* the transport demand, F^R, relevant to the reference link can also be calculated, either by use of equations analogous to **[23]** or by use of Equation **[21a]** and **[16a],** to obtain, given the number r_R of routes inherent in the reference link,

[25] $$F^R = \Omega\, \Sigma_i \exp(Q^R/x_i^R), \qquad (i = 1, 2, \ldots, r_R)$$

in which $\Omega = 1 / \exp(1+1/k)$ **.**

Interesting to note, network constant $\Omega = 1/e^{1+1/k}$ has the physical dimensions of a traffic flow, whose value depends on *k,* i. e., on the measurement system adopted for transportation costs and traffic flows. The value of network constant Ω is usually very small, but it grows exponentially with the value of network constant *k.* The latter constant has been introduced and mathematically defined as a co-factor that stresses the influence of the traffic stream intensities on the formation of the users' choice or demand for travel routes.

In this connection, it's worth remembering that the values of all network/link constants *k, u,* and Q^L, depend on the measurement system adopted for quantifying transportation costs and traffic streams.

3.2.1* –** As a logical consequence of what stated in the preceding paragraphs, the **overall transport demand *F* in the network is not a *datum of the model.

F can be identified only after the use of Equation **[25],** through **[24],** and – possibly – after using equations like **[23]**, depending on what of the required input is available.

This is a point to be borne in mind prior to applying the model, to avoid the introduction of non-required input that could bring about logical inconsistencies.

3.3 – Actual Distribution of the Traffic Streams

As to the aggregate traffic, the second step of the simulation process consists in the determination of the traffic load that may actually be expected on each route of the system.

This aspect of the problem is particularly important in the analysis of the road network, where the traffic is not kept under a centralized planning control. This is one of the main causes of frequent traffic congestions in sections of the road network, where the flocking users (often unaware of the actual traffic conditions they are going to encounter) do not only cause the saturation of the road capacity but also an over-concentration of vehicles that lowers the flow below or much below the road capacity.

Therefore, the analysis of road traffic – because of the relatively higher degree of unpredictability of road traffic – involves general concepts that can be applied, where appropriate and with due adjustments, also to other transportation modes.

In the application of the model, concepts such as "arc", "link", "route", "capacity", "saturation", "congestion", "flow speed", etc., clearly defined for road routes in the subsequent paragraphs, must carefully be defined and appropriately managed – case by case – as to other transportation modes, such as railways, waterways, etc., according to the physical and functional characteristics that are proper to the study system.

The necessary data relevant to alternative non-road transportation modes are heavily depending on the operative plans of the principal traffic managers of any such particular transportation infrastructure.

Dealing with railway, for example, an *over-concentration* of passenger load in trains (many/most passengers are compelled to stand during their travel) may not necessarily imply *flow congestion* and even result, instead, in a higher flow of passengers. Railway congestions appear in particular forms of *over-demand*, when transport dysfunctions and relevant travel delays are connected with extraordinary attempts of the transport offer to meet an exceptional transport demand.

3.3.1 - Assuming the traffic on any *road* network as an adequate basis for the analysis, the theoretical principles to be adopted are related to the fundamental equation of vehicular flows, as expressed by

[26] $F = DV$, or $D = F/V$, or $V = F/D$,

in which:

F is the stream **intensity**, as given by the number of transportation units (*tu*) crossing a conventional road section in a given time unit;

D is the mean vehicular **density** in transportation units per road-length unit,

and V is the mean stream **speed** in road-length units per time unit.

Without entering detailed definition for the transportation units, it is only assumed – for the analysis purposes – that the physical dimension of "length" may be associated with each of them. Such physical quantity (*length*) is not a constant quantity, since it depends of the characteristics of the traffic flow, and **varies** with the stream's speed from an average conventional minimum b (body length), when the speed is nil, to a maximum length that depends on the maximum speed the transportation unit is allowed to attain on the road considered.

Therefore, according to a hypothesis largely shared by road transportation engineers, a conventional *transportation unit* l can be expressed by the following function of its speed:

[27] $l = b + aV^2$,

in which coefficient a depends basically on the road characteristics.

A different definition regards the conventional *transport load unit*, which is commonly indicated by the well-known acronym "*pcu*" (passenger car unit).

This model uses both the definitions above, according to the analysis or to the simulation circumstances involved. Definition **[27]** is particularly used to determine the values for F, D and V by use of equations **[26]** in terms of *vehicles per time unit*.

Acronym *pcu* is instead currently used to express the value of traffic stream intensity in terms of *pcu per time unit*, and the value of stream density in terms of *pcu per road-length unit*.

The conversion of *vehicular units* into *load units,* and *vice-versa,* depends on the engineering criteria adopted for the particular study case.

If h denotes the *road-length unit*, the value of stream density can in particular be expressed in *vehicles per road-length unit* by

[28] $D = h/l = h/(b+aV^2)$

Because of this definition, *stream density* – being the ratio of a length to a length – has no physical dimension. Instead, coefficient a has the physical dimension of the inverse of an acceleration.

To be noted: In this analysis, for simplification purposes, the road-length unit h relates to any *one-lane* road.

If the road lanes considered for each stream direction are more than one, h must be multiplied by the number of lanes.

By substitution of definitions **[27]** and **[28]** in Equations **[26],** and by subsequent algebraic manipulations, the following set of relations is obtained:

[29] $F = hV/(b+aV^2)$,

to express stream intensity as a function of the average speed; and

[30] $V = [h \pm (h^2 - 4abF^2)^{1/2}]/2aF$,

to express the stream mean speed as a function of the relevant stream intensity;

and also:

[31] $V = [(h - bD)/aD]^{1/2}$,

to express the stream mean speed as a direct function of the stream density; and

[32] $F = [D(h - bD)/a]^{1/2}$,

to express the stream as a function of its density; and

[33] $D = [h \pm (h^2 - 4abF^2)^{1/2}]/2b$,

to express the stream density as a function of the stream intensity.

It is important to observe that the sign "$\pm$" in Equations **[30]** and **[33]** implies different values of the *dependent* variables (i.e., V and D, respectively) with respect to those values of the *independent* variables (i.e., D and F, respectively) which are associated with the saturation of the road capacity. These particular values of the *independent* variables split the relevant functions into two different sections: One section relates to *non-congested (ordinary) traffic condition*, whereas the other one relates to *congested traffic condition.*

In Equation **[30]** sign "+" after h gives the mean flow velocity values in *ordinary traffic condition*, whereas sign "+" after h in Equation **[33]** gives the average vehicular densities in *congested traffic condition*.

It's easy to show that the calculation of the maximum intensity of the stream depends on the values that parameters a and b must be given in the study system. It's sufficient to derivate **[29]** with respect to speed V and to equal this derivative to zero. To obtain:

$$\partial F/\partial V \; = \; h(b - aV^{2}) \,/\, (\, b + aV^{2})^{2} \; = \; 0 \,,$$

whence

$$V \; = \; \sqrt{b/a} \,.$$

By use of this value for V in **[29],** one gets *the maximum stream intensity* as expressed by

[34] $$F_{max} \; = \; h \,/\, 2\sqrt{ab} \;,$$

which is the stream intensity that saturates the road capacity**;** while $V = (b/a)^{0.5}$ is the flow speed proper to the saturation stream.

Considering that $D_{max} = h/b$ is the *absolute maximum vehicular density*, which implies $F_0 = 0$ (i.e., the stoppage of the vehicular stream on the road),[3] Equation **[33],** after accounting for **[34],** shows that the vehicular density at the road capacity saturation is precisely half of the absolute maximum density, i.e.,

$$D_{sat} \; = \; h/2b \,.$$

Annexed Tables give a visual understanding of the above mathematical findings, namely:

Table I is a synoptic list of the interrelated functions, including a table that shows how vehicular flow and speed may depend on various combinations of a and b.

Table II gives diagrams of flows in function of speed.

Table III is a diagram of flows in function of vehicular density.

Table II, in particular, summarizes the basic physical laws that govern the vehicular flows on roads. The diagram shows how the vehicular speed, intensity and density change in roads having different number of lanes.

It's worth noting, in Table II, that the trigonometric tangent of the angle in Ô of the triangles drawn in white color is the vehicular *density* in the lanes, because vehicular density is the ratio of the traffic flow intensity F to the flow mean speed V.

Furthermore, this table displays the two different traffic regimes that are caused by vehicular density. The curve that describes the flow intensity in relation to the flow mean speed (in terms of traffic units flowing in an hour) has an apex where the flow equals the road capacity. The traffic regime described by the section of the curve at the right hand side of the apex describes the traffic flow in *ordinary regime,* whereas the section of the curve at the left hand side of the apex describes the traffic condition in *congestion regime.*

(The tables, together with the diagrams and the subsequent sections of the text, are continued in the pages that follow)

- [3] If $D = h/b$ in Equation **[32],** then $F = 0$.

TABLE I

BASIC RELATIONSHIPS USED FOR DETERMINING THE EXPECTED STREAM ASSIGNMENT (and for assessing travel duration times)

(0) Basic equations: $F = DV$; $D = F./V$; $V = F/D$;
in which:

F is the traffic stream, in transport units per hour (*tu/h*);
V is the average stream speed - in *km/h* - in the covered section considered;
D is the average stream density - in *tu/km* - in the same section.

(1) Definition of safety distance S between subsequent vehicles in a vehicle-train motion: $S = aV^2$;

for which, on a multilane carriageway, it's $a = 1/160$ and,
on a one-way lane of simple two-lane roads, it's $a' = 1/122.5$. Hence: $S' = a' V^2$.

(2) Definition of stream density D in relation to safety distance S (or S'):
$D = h/(b + S) = h/(b + aV^2)$;
or $D = h/(b + S') = h/(b + a' V^2)$;
in which b is the vehicle average length (transport unit, *tu*).

(3) From (2) above, flow F may be expressed as a function of average stream speed V:
$F = hV/(b + aV^2)$.

(4) From previous (3), speed V can also be expressed as a function of stream F:
$V = [h + (h^2 - 4abF^2)^{1/2}] / 2aF$.
From this second degree equation two speed values are obtained for every single stream figure.
It is, in fact, the inverse equation of (3) as represented in TABLE II, for a multilane carriageway.
As presented here, the relevant diagram appears similar to the analogous empirical flow diagrams of the usual traffic handbooks.

(5) Average stream speed V as a direct function of stream density D:
$V = [(h - bD)/aD]^{1/2}$.

(6) Stream F as a direct function of stream density D:
$F = [D(h - bD)/a]^{1/2}$.

(7) Stream density D as a function of stream intensity F (the inverse of previous (6)):
$D = [h + (h^2 - 4abF^2)^{1/2}] / 2b$.
In order to obtain real values from the above equation - as well as from equation (4) - it must always be $F < h / 2(ab)^{1/2}$, i.e.: F must always be smaller than - or equal to - the saturation (maximum) stream that is specific of the lane considered.

The value of parameter h is 1000 *m* in the metric system, and varies with the measurement system adopted.
The conventional values assigned by the table below to parameters a and b may vary according to the specific regional road system under study.

STREAM AND SPEED VALUES IN LANE SATURATION UNDER DIFFERENT ROAD CONDITIONS

b	*(more than one lane per direction)*			*(one lane per direction)*		
	a =1/160	$F_{max} = 500/(ab)^{1/2}$	$V = (b/a)^{1/2}$	a' =1/122.5	$F_{max} = 500/(ab)^{1/2}$	$V = (b/a')^{1/2}$
8.5	0.00625	2169	37	0.0081633	1898	32
9	0.00625	2108	38	0.0081633	1845	33
9.5	0.00625	2052	39	0.0081633	1795	34
10	0.00625	2000	40	0.0081633	1750	35
10.5	0.00625	1952	41	0.0081633	1708	36
11	0.00625	1907	42	0.0081633	1669	37
11.5	0.00625	1865	43	0.0081633	1632	38
12	0.00625	1826	44	0.0081633	1598	38
12.5	0.00625	1789	45	0.0081633	1565	39
13	0.00625	1754	46	0.0081633	1535	40

Note: parameter b (which includes the average safety distance kept in a situation of traffic stoppage and steady queue) depends upon the vehicle mix in the traffic stream.

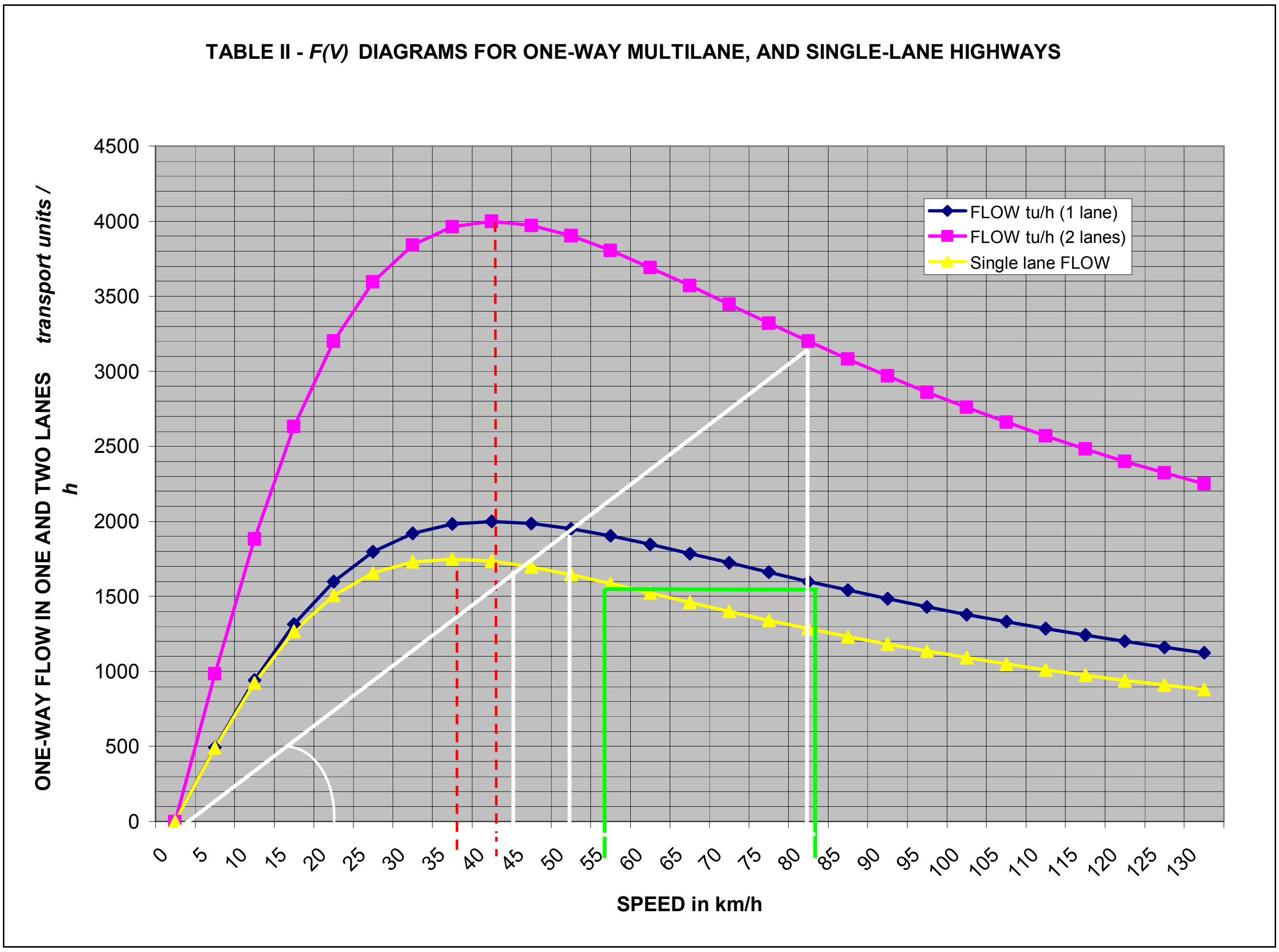


TABLE II - *F(V)* DIAGRAMS FOR ONE-WAY MULTILANE, AND SINGLE-LANE HIGHWAYS
ONE-WAY FLOW IN ONE AND TWO LANES *transport units / h*
SPEED in km/h
FLOW tu/h (1 lane)
FLOW tu/h (2 lanes)
Single lane FLOW
0
500
1000
1500
2000
2500
3000
3500
4000
4500
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110 115 120 125 130

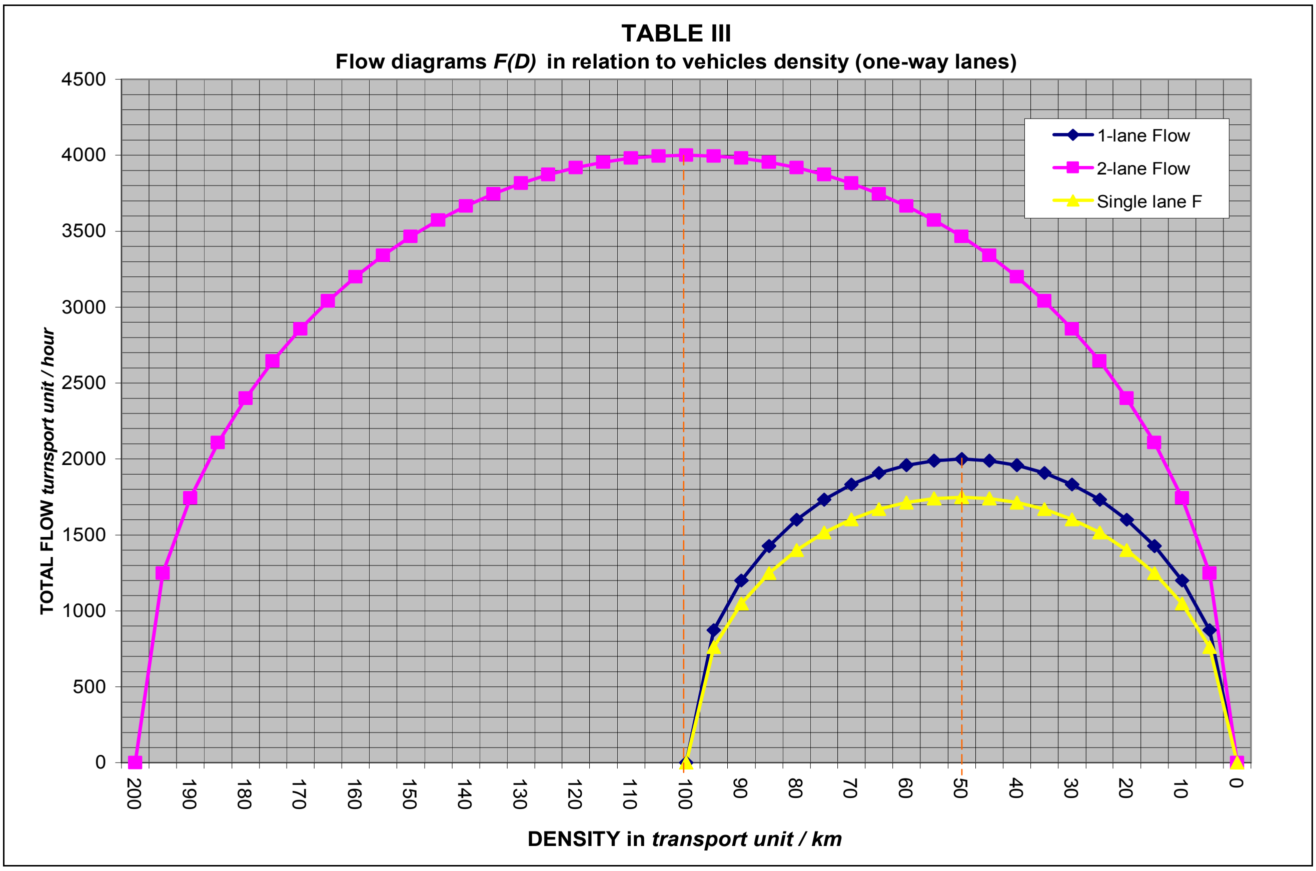


TABLE III
Flow diagrams *F(D)* in relation to vehicles density (one-way lanes)
TOTAL FLOW *turnsport unit / hour*
DENSITY in *transport unit / km*
1-lane Flow
2-lane Flow
Single lane F
0
500
1000
1500
2000
2500
3000
3500
4000
4500
200 190 180 170 160 150 140 130 120 110 100 90 80 70 60 50 40 30 20 10 0

3.4 – Streams in Road Congestion

The overall traffic conditions in the road network may cause the travel demand for some oriented arcs to exceed the capacity of these arcs.

The excess in the travel demand results in a concentration of vehicular units greater than the maximum stream intensity allowed by the capacity of the arc. As a consequence, the vehicular flow in the arc declines to intensity levels that are inevitably lower than both capacity and demand. The road arc is in a congestion condition.

It is therefore reasonable to assume that the degree of congestion in the arc depends on the level of the *excess demand.*

The analytical function, which expresses the *vehicular density* in any congested road arc ***x*** as depending on the excess demand, must be subjected to a number of constraints.

If d_x is the flow demand for arc x, and ϕ_x is the arc capacity, then $\Delta_x = d_x - \phi_x > 0$ represents the *excess* in the flow demand. [4]

Congestion density D_x in the arc is a function of Δ_x subject to the following constraint:

[35] $$0 \leq \Delta_x \leq \phi_x \quad ;$$

This constraint means that in no case the *excess demand* is greater than the arc capacity, because the vehicular *density* (see Equations **[33]** and **[34]**) has *real* numerical values only if the relevant flow intensity is between zero and ϕ_x.

In simpler terms, the number of vehicles that enter the arc cannot physically exceed the number of vehicles that determines the maximum density D_x^* in the same arc.

As a function of Δ_x , density D_x in the arc must be

[36] $$\underline{D}_x = D_x^*/2, \quad \text{if} \quad \Delta_x = 0 \;;$$

Whereas it is $D_x = D_x^*$ when $\Delta_x = \phi_x$.

Continuous functions Δ_x and D_x that match constraints **[34]** and **[35]** are the following:

[37] $$\Delta_x = \phi_x \mathbf{cos}^{\mathbf{2}}\mathbf{(}\pi D_x/D_x^*\mathbf{)},$$

and its inverse

[38] $$D_x = (D_x^*/\pi)\mathbf{arcos[-(}\Delta_x / \phi_x)^{1/2}\mathbf{]}.$$

If, in the congested arc, this is the density in function of excess demand Δ_x , then the *actual corresponding stream intensity* is obtained by substitution of D with D_x in Equation **[32]**, to express *congested flow* f_x as follows:

[39] $$f_x = \mathbf{[}\, D_x(h - bD_x) / a\mathbf{]}^{1/2} .$$

[4] It should go without saying that $\Delta_x \leq 0$, when $d_x \leq \phi_x$, represents instead the *balance* to the arc saturation.

As a consequence, the congested stream mean speed is expressed by

[40] $$V_x = f_x / D_x = \mathbf{[}(h - bD_x) / aD_x\mathbf{]}^{1/2} \;;$$

whence

[41] $$t_x = L_x D_x / f_x$$

is on a average the time spent by a vehicle to travel over the congested arc x of length L_x .

The observation of real congestion proves that permanent traffic flow stoppages because of congestion never occur, since the *apparent* stoppage does sooner or later cease, and all the involved transportation units reach their destinations. This forces to assume always $D_x < D_x^*$, or $V_x > 0$ (i.e., $t_x < \infty$), especially considering that all the quantities introduced by the analysis are taken as *mean values*, which must by hypothesis satisfy the given travel demands within the conventional time unit.

3.5 – Flow Compression and Reduction

It is generally assumed that any traffic flow that causes congestion is the summation of different streams that run along different routes but have the congested arc in common.

The sum of these stream densities results in congestion flow f_x as defined by Equation **[39]** above.

Difference

[42] $$B_x = d_x - f_x$$

is the amount of the flow demanded for x that has been *reduced* into *compressed flow* f_x . Actually, the "compression" regards the flow density, though *compressed* density may be taken as a measurement of the relevant *flow potential*. Normally, in fact, as soon as the cause of the congestion ceases, the various streams that have been reduced/compressed by congestion can regain their respective initial values.

Demand d_x for arc x, brought about by the bundle of different routes that have the congested arc x in common, may be considered – in terms of probability – as composed by a set of different *sub-demands* d_x^r translated into *compressed* streams f_x^r as expressed by

[43] $$f_x^r = (d_x^r / d_x)\, f_x ,$$

in which r indicates any route of the bundle.

The initial values of the streams, which have been compressed by the congestion in x, can be restored, if the road arcs subsequent to congested link x provide the physical conditions that allow the *re-expansion* of the compressed streams.

Suppose that any compressed stream, whose initial value is f^r, must regain its proper mean velocity V_r to re-expand completely in running along the arc subsequent to congested arc x. In the congested arc, all compressed streams have the same mean speed V_x, as given by Equation **[40]**.

If g is the mean acceleration associated with any transportation unit of the compressed stream f_x^r getting out from x, then

[44] $$t = (V_r - V_x)/g$$

is the time taken by all the units of the re-expanding stream to re-attain their original density and speed. This implies that length L^r_{x+1} of the link after x must be at least

[45] $$L^r_{x+1} = (V_r^{\,2} - V_x^{\,2}) / 2g.$$

As a conclusion, any stream incurring congestion in any arc x of its route is not suppressed, if it can re-expand in a subsequent arc whose length is greater than **[45].**

Experience indicates that compressed streams can restore their original intensities in a few thousand meters, however low the congestion speed may be.

Nevertheless, some particular cases must be allowed for.

3.6 – Special Situations

For whatever reason, it may happen - during the application of the model for simulation purposes - that some compressed streams cannot re-expand after congestion.

From the theoretical standpoint, two cases are possible, in which streams cannot overcome congestion:

(a) excess demand Δ_x for arc x is greater than capacity ϕ_x of the arc;

(b) excess demand Δ_x for arc x is less than capacity ϕ_x of the arc, but some of the streams there compressed have no further possibility of re-expansion.

Case (a):

Occurs when arc x has attained a *flow stoppage state*, and no additional transportation unit can physically enter the clogged arc.

Should statistics show that such a case is very frequent for any arc x, then the model must avail itself of the following **additional hypothesis:**

"*A relatively great number of network users are acquainted with the risk of traffic stoppage in arc x*".

This hypothesis implies that, before including risky arcs in the routes they choose, many users systematically tend to opt for alternative routes, which exclude too risky arcs.

In operational terms, the simulation addresses this case by calculation of the portion of rejected demand for arc x on the basis of a conventional minimum average speed tolerated in the congested or quasi-blocked arc. By use of Equation **[29]** it is possible to assess the intensity of the flow associated with the conventional minimum speed, and by Equation **[42]** it is possible to determine the portion of the demanded flow that is rejected by the clogged arc.

Let $\Phi_{ix}^{\,L}$ be the portion of stream f_i^L that is "rejected" by clogged arc x, and j indicate any of those alternative routes of link L which do not include clogged arcs. Then, stream portion $\Phi_{ix}^{\,L}$ is supposed to be re-distributed according to sub-portions $\vartheta_{ix}^{\,L}$ expressed by

[46] $$\vartheta_{ix}^{\,L} = \Phi_{ix}^{\,L} \exp(Q_i^L/x_i^L) \,/\, \Sigma_j \exp(Q_j^L/x_j^L),$$

in which $\Sigma_j \vartheta_{jx}^{\,L} = \Phi_{ix}^{\,L}$.

Case (b):

Is the case in which the *impossibility of re-expansion* for any stream compressed by congestion results from the insufficient length of the arc subsequent to the congested one; or it results from that the subsequent arc, though sufficiently long, enables the vehicles to quickly achieve an average speed that is much higher than that proper to the original sub-flow. As already considered, in fact, flow intensity declines with the average vehicular speed allowed by the road.

It doesn't seem reasonable to assume that this is a frequent case, considering the statistical evidence provided by real traffic situations. The case, however, is possible, and deserves some theoretical attention.

According to the various conditions that may affect simulation exercises, this kind of problem can be overcome in different ways.

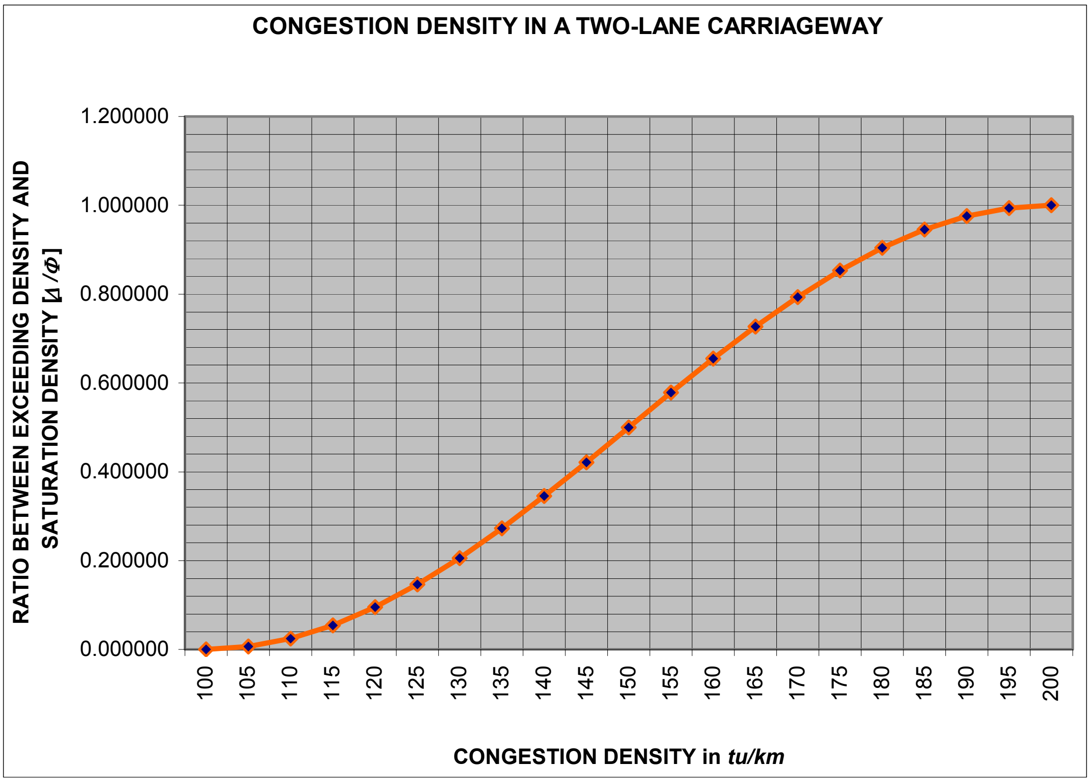
In general, the route to consider is formed by a sequence of several arcs, which sooner or later allow the restoration of the streams existing before being compressed by congestion. Thus, it's worth extending the analysis to the whole set of arcs that follow the congested one, to establish whether and where the solution is found.

More critical is the case in which the route sections – subsequent to the congested arc – indicate no possibility of re-expansion for the compressed streams. In this extreme case, the most reasonable solution is to make the problem identical to that presented by portions of flow demand rejected by clogged arcs.

It is likely, however, that other solutions are possible also in connection with particular infrastructure characteristics of the destination pole, by which the stream re-expansion may eventually be attained *within the destination road infrastructure,* despite the unfavorable sequence of arcs preceding the destination pole.

Annexed **Table IV** shows a diagram of the exceeding densities in a congestion state.

TABLE IV



4 – THE *DISAGGREGATE* MODEL

4.1 - General

The transportation flows addressed in the preceding sections include any category of both vehicles and transported loads. The solution to the problem, however, is usually requested separately for either vehicle or load category. In particular, the two main load categories to consider consist of *passengers* (i.e., human travelers) and *goods*.

This section of the paper provides a solution to the problem of assessing the *probable demand for routes* relative to different categories of traveling loads.

The conceptual approach to this new problem is substantially the same as before, though a few additional aspects must be pointed out, which involve some changes in the mathematical procedure.

In the case of the "aggregate" model, any network user, prior to selecting the route to use, is supposed to estimate the impact of aggregate transport streams on the particular mean *aggregate* unit cost associated with the traffic stream that the same user is contributing to create.

In addressing *disaggregate* streams, any network user is still supposed to estimate the impact of *aggregate* transport streams, but the estimated impact is now on the mean unit cost of the *particular category stream* that concerns the user.

There is no cost homogeneity between aggregate and disaggregate transport stream, as the latter is only one of the different non-separable components of the former. Particular transportation mean unit costs are associated with each category of *disaggregate stream*, whereas the estimated unit cost relevant to *aggregate* flows may be expressed by any appropriate average of various category unit costs. One major implication of this for the analysis is that the influence of the aggregate stream intensity on each category of transport must be determined through a procedure in which no strictly logical relationship can *a priori* be established between aggregate and disaggregate transportation unit costs.

For each route considered, the *aggregate stream is then taken as a datum* that contributes to the determination of the *transportation category mean unit cost* assessed by the category user.

Therefore, the access to the *disaggregate model* requires an initial application of the aggregate model.

4.2 – The Approach Adopted

The model intends to formulate the probable route demand expected from *category users*, such as, for example, passenger or commodity transporters.

The problem analysis summarized below regards an unspecified category of users.

All that which has been considered as affecting the user's decision in the aggregate model remains valid for the disaggregated model, so that it is possible to refer immediately to Equation **[4]**, to re-write this – in the disaggregate context ─ in the following form:

[47] $\Gamma_{\alpha i}{}^{L} = c_{\alpha i}{}^{L}\,(1+k\mathbf{Ln}\,d_i^L)\,g_{\alpha i}{}^{L} = J_{\alpha i}{}^{L}\,g_{\alpha i}{}^{L}, \qquad \forall \alpha, i, L,$

in which:

- $J_{\alpha i}{}^{L} = c_{\alpha i}{}^{L}\,(1+k\mathbf{Ln}\,d_i^L)$ is the *conditioned mean unit cost* of the α-category load transported on route *i* of link *L*, as estimated by category users with respect to any route *i* of link *L*;
- $\Gamma_{\alpha i}{}^{L}$ is the overall cost of *probable* α-category stream $g_{\alpha i}{}^{L}$ per time unit;
- $c_{\alpha i}{}^{L}$ is the 'free-route" – or *technical* mean unit transportation cost relevant to the α-category considered, as estimated by users in absence of the traffic effects.

All these costs are known, since they belong to the input data set;

- $k > 0$ is the system constant multiplier introduced by Equation **[18]** of the aggregate model;
- d_i^L is the route aggregate demand determined through previous application of the aggregate model. It is a *given* input necessary to start the application of the disaggregate model;
- $(1 + k\ \mathbf{Ln}\ d_i^L)$ is the "*conditioning factor*" already introduced by the aggregate model.

It must be pointed out that the measurement system for disaggregate streams $g_{\alpha i}{}^{L}$ shall be made homogeneous to the measurement system adopted for the aggregate streams.

Still adopting the analytical procedure followed to build the aggregate model, and in an analogy with definition **[7]**, it is possible to use Equation **[47]** above to express the *category* net benefit U_α that is associated - in the conventional time unit - with the category transportation activity carried out in the whole network:

[48] $U_\alpha = \mathbf{\Sigma}_L\, U_\alpha{}^{L} = \mathbf{\Sigma}_L \mathbf{\Sigma}_i\, [B_\alpha{}^{L} - c_{\alpha i}{}^{L}(1 + k\,\mathbf{Ln}\,d_i^L\ )]g_{\alpha i}{}^{L}.$

It must be stressed that this U_α, as well as symbols $U_\alpha{}^L$, $B_\alpha{}^L$, $c_{\alpha i}{}^L$, and $g_{\alpha i}{}^L$ in definition **[48],** are affected by index α to identify only one particular transportation category addressed (any one of the several possible ones). This remark draws attention to that the quantity defined by **[48]** is only one *component* of quantity **[7].**

For the formulation of the disaggregate model, *it is provisionally assumed* that every category transport demand $G_\alpha{}^L$ - whatever α - is given for each link *L*.

Therefore, in a close analogy with equations **[6]** of the aggregate model, the following *N*(*N*-1) equations can be written for any transportation category :

[49] $\mathbf{\Sigma}_i\, g_{\alpha i}{}^{L} = G_\alpha{}^{L}, \qquad \forall \alpha, i, L, \qquad \{L = 1,...N(N\text{-}1)\}$ **.**

From this, the overall demand for α-category transportation in the network results in the following definition:

[50] $\mathbf{\Sigma}_L\, \mathbf{\Sigma}_i\, g_{\alpha i}{}^{L} = \mathbf{\Sigma}_T\, G_\alpha{}^{L} = G_\alpha\,,$

Then, it is possible to define the *probability of category transportation demand* relevant to any link *L* by the following ratio:

[51] $q_{\alpha i}{}^{L} = g_{\alpha i}{}^{L}/G_\alpha\,, \qquad \forall \alpha, i, L.$

This definition, as previously for p_i^L in the aggregate model, implies the possibility of identifying a *probability distribution* for the transportation demands relevant to each category, with the relative analytical advantages.

For each probability distribution (one for each category) there is the relative *probability uncertainty* E_α . These uncertainties are mathematically constrained by the information expressed by respective Equations **[48]** and **[49]**, once symbol $g_{\alpha i}{}^L$ in these has been replaced by definition $G_\alpha\, q_{\alpha i}{}^L$ obtained from **[51].**

The solution to the new problem, i.e., the determination of the values for all $q_{\alpha i}{}^L$ (and, therefore, also for all $g_{\alpha i}{}^L$), consists – as it was for the aggregate model ─ in the determination of the constrained maximums of all uncertainty functions E_α.

Appealing once again to the method of Lagrange multipliers, the solutions to the problem are now given by:

[52] $q_{\alpha i}{}^{L} = H_\alpha{}^{L}\,\mathbf{exp(}\text{-}\lambda_\alpha J_{\alpha i}{}^{L}\,\mathbf{)\ /\ \Sigma}_i\,\mathbf{exp(}\text{-}\lambda_\alpha J_{\alpha i}{}^{L}\,\mathbf{)},$

in which

[52.1] λ_α is a *category constant* for the entire network;

[52.2] $H_\alpha{}^{L} = G_\alpha{}^{L}/G_\alpha\,;$

[52.3] $J_{\alpha i}{}^{L} = c_{\alpha i}{}^{L}(1 + k\,\mathbf{Ln}\,d_i^L)\,.$

To determine *category* constant λ_α, it is sufficient to know, for each category, the ratio between two category streams (or the respective probabilities) relevant to two different routes of the same reference link *R*. This is evidenced by the following equation:

[53] $\lambda_\alpha = \mathbf{Ln}(q_{\alpha m}{}^{R}\,/\,q_{\alpha n}{}^{R})/(\,J_{\alpha n}{}^{R}\,\text{-}\,J_{\alpha m}{}^{R}\,)\,;$

which is derived from **[52]** applied once for $q_{\alpha m}{}^R$ and once for $q_{\alpha n}{}^R$.

It must be remembered that all the category streams that form the two reference aggregate streams f_m^R and f_n^R are known by hypothesis. Thus

[54] $f_m^R = \mathbf{\Sigma}_\alpha\, g_{\alpha m}{}^{R}\,,$ and $f_n^R = \mathbf{\Sigma}_\alpha\, g_{\alpha n}{}^{R}\,.$

Which means that all the network category constants λ_α can be calculated on the basis of the given reference category streams.

4.3 – Calculation of Disaggregate Demands

In the preceding paragraph, transport category demand $G_\alpha{}^L$, concerning any link *L,* was *provisionally* supposed to be a given input. Actually, if one considers the *conditioning factor* as a significant connection between aggregate and disaggregate model, then the previous calculation of the **aggregate** route demands (these to be turned into aggregate *streams* only after the congestion analysis) provides sufficient additional input for considering also *disaggre-gate demands for link* **and** *route demands* **as calculable unknowns**.

In view of this, it is necessary to suitably re-write Equations **[52],** in order to point out that these equations do now allow for the unknown pairs $g_{\alpha i}{}^{L}$ **and** $G_{\alpha}{}^{L}$, instead of $g_{\alpha i}{}^{L}$ only. Thus, after defining the ratio between the exponential coefficients in **[52]** as

[55] $\omega_{\alpha i}{}^{L} = \exp(-\lambda_{\alpha} J_{\alpha i}{}^{L}) \,/\, \Sigma_i \exp(-\lambda_{\alpha} J_{\alpha i}{}^{L}), \qquad \forall \alpha, i, L,$

and taking into account definitions **[51]** and **[52.$_2$]**, Equations **[52]** become

[52a] $\omega_{\alpha i}{}^{L}\, G_{\alpha}{}^{L} - g_{\alpha i}{}^{L} = 0\,, \qquad \forall \alpha, i, L.$

The number of equations of this kind is equal to the number r of the routes of link L, whatever r and L.

It's important to bear in mind that the sum of coefficients $\omega_{\alpha i}{}^{L}$, which are known by hypothesis, each of them concerning category α on route i of link L, must equal 1. Which means

[56] $\sum_{i=1}^{r} \omega_{\alpha i}{}^{L} = 1\,, \qquad \forall r, \alpha, L,$

r being the number of routes inherent in L.

The other set of equations, which must be combined with Equations **[52a]** to define the complete system of equations involved, is provided by

[57] $\Sigma_{\alpha}\, g_{\alpha i}{}^{L} = d_i{}^{L}\,, \qquad \forall i, L,$

to simply express that the sum of the demand for route i from the various category users must equal the aggregate demand for the same route, as previously determined by application of the aggregate model.

One additional equation, which may be used – should necessity arise – is, for every link,

[58] $\Sigma_{\alpha}\, G_{\alpha}{}^{L} = F^{L}, \qquad \forall L,$

which imposes that the sum of all the transport category demands for link L equals the aggregate demand F^L for the same link, the latter demand belonging either to the inputs or to the outputs of the aggregate model.

For these equation systems in the unknowns $g_{\alpha i}{}^{L}$ and $G_{\alpha}{}^{L}$, the known terms are provided by the previously calculated *aggregate demands for route* relevant to link L , as regarded by each equation system.

It may sometimes be necessary to introduce the aggregate transport demand as a known term for one or more links, when the coefficient matrix of the equation system requires so, because of the need to substitute any equations whose coefficients are linear combinations of other equation coefficients.

In practice, the possible configurations of equation systems of this kind depend on the number of transport categories and link routes regarded by the equations.

Therefore, there is in general to consider that number of equations and set of coefficients (including known terms) vary from link to link of the study network.

Two Excel sheets annexed to subsequent Paragraph **6** give a couple of examples of how the coefficient matrices of such equation systems can be built according to different combinations of number of categories and number of routes per link.

5 - COST CALIBRATION

In the application of the model to real cases, a preliminary overall calibration exercise is normally needed, with a view to making all the transportation unit costs mutually consistent and compatible with the logic structure of the model.

This necessity is impellent when the application involves the use of the disaggregate model in conjunction with the aggregate model.

The combined model application requires that the aggregate technical unit costs be assessed in conside-ration of the relevant disaggregate technical unit costs, though there is no strict logical connection between the two sets of input costs.

The criteria for assessing the values of the input data depend essentially on reliable statistics and on specific professional experience.

From the theoretical point of view, for example, the calculation of aggregate unit costs through weighed means of disaggregate costs is not advisable, if the weights consist of data that the majority of the transportation system's users are not supposed to know.

It should be born in mind that the simulation model is an attempt at reproducing – according to a presumable average behavior – the reasoning followed by any reasonable user that makes choices based on common experience and information. There are obviously transport operators that can dispose of a higher and uncommon amount of data and infor-mation. However, the decisions of such operators – especially of those who manage railways, waterways and the like – can to a major degree affect the functionality of special infrastructure of the system, but – if it is road transportation – do not affect the behavior of the determinant part of the network users.

In any case, also the majority of the users of any special transportation infrastructure cannot always avail themselves of an adequate amount of reliable information, so that the actual transport offer may not match the transport options demanded by the users.

In assessing aggregate costs, model operators might be tempted, for example, to use disaggregate transport *demands on links* as weights for weighed means of disaggregate costs. The disaggregate transport demands on links are considered as part of the unknowns of the model, and they are certainly unknown to almost all the users of any transportation system. Of course, there may be transport specialists or operators that can exceptionally attain such level of information, but the use of unlikely data of that kind would be inappropriate in the light of the model's theoretical approach to the problem.

Moreover, weighed means like those mentioned above would not substantially improve the quality of the technical assessment of the input data, for no strict logical criterion can justify the use of such weighed means.

What matters more to optimize the model's performance is to achieve a good approximation in assessing the *relative* cost of traveling on different routes as it's on an average perceived by different users: This means to establish that – whatever the link – the traveling cost associated with route i is, for instance, 1.3 times the cost associated with route j, while 0.8 times the cost associated with route r ; and so on.

The principal information source is provided by direct traffic surveys. Accurate surveys are very expensive and can usually be conducted only on very few links and routes of an inter-regional transportation network.

As clearly expressed by the hypotheses on which the model's theoretical framework has been constructed, adequate traffic surveys must at least be conducted on two different routes of one of the significant links of the study system.

Once purpose, aims and objectives of the particular transportation study have been clearly identified, also the survey extent, methodology and degree of accuracy can be established.

5.1 – Calibration by Inverse Use of the Model

The data coming from surveys can be used for an *inverse use* of both the aggregate and of the disaggregate model, i.e., using traffic demands and/or actual traffic streams as inputs, and aggregate and/or disaggregate costs as unknowns.

How this is actually possible doesn't need much explanation**:** All the formulas introduced above make this possibility almost self-evident. What is important**:** It's always necessary to start from a "reference link", in order to determine the network constants u and k, which are indispensable for the calibration of both aggregate and disaggregate costs.

It is worth pointing out that **the costs to be calibrated are the "free-route costs" (or "technical costs") and not the *conditioned actual costs*** that direct traffic and statistical investigations can normally provide. Both the aggregate and the disaggregate model are based on "free-route costs", in the consideration that a useful application of the model should be significantly possible also in absence or in a serious lack of data from direct surveys and investigations.

5.1.1 – Calibration of Aggregate Costs

Given all the streams of the "reference link", the determination of network constants u and k is possible if the aggregate technical costs relevant to any two routes of this link are also given. As seen in Paragraph **3**, these conditions are sufficient for the use of formulas **[18], [19]** and **[21],** by which the calculation of k and u is possible. Once these constants have been determined, the cost calibration can be carried out link by link.

As to any other link, the calibration of the aggregate unit costs must start from the assessment of one of them, relevant to any route of the link. Let's suppose that *technical unit cost* C_i^L, relative to route i of link L, has been assessed according to usual specific criteria. It's easy to show that every other technical cost associated with every other route of the link remains univocally determined by use of the relevant aggregate stream, which is now known by hypothesis. It is also assumed that – as to the calibration – the traffic streams coincide with the actual route demands for transport.

From Equation **[16]**, and accounting for **[20]**, we know that

[59] $$p_i^L/A = f_i^L \mathbf{e}^{1+1/k} = \mathbf{exp}[Q^L/(1+uC_i^L)], \quad (\forall\ i, L),$$

from which – either for i or for any other route x of the link – it is possible to isolate link constant Q^L as follows:

[60.a] $$Q^L = (1+uC_i^L)\mathbf{Ln}(f_i^L \mathbf{e}^{1+1/k})$$

[60.b] $$Q^L = (1+uC_x^L)\mathbf{Ln}(f_x^L \mathbf{e}^{1+1/k}).$$

Equaling the right hand sides of these two equations leads to

[61] $$C_x^L = [(1+uC_i^L)\mathbf{Ln}(f_i^L \mathbf{e}^{1+1/k})/\ u\mathbf{Ln}(f_x^L \mathbf{e}^{1+1/k})] - 1/u,$$

C_x^L being the cost to be determined, whatever x of L.

5.1.2 – Calibration of Disaggregate Costs

The disaggregate model, at variance with the aggregate one, makes explicit use of *conditioned costs* $J_{\alpha i}^L$ in the formulas that express the disaggregate transportation demand (category stream) on each route of the transportation network. (Instead, in the aggregate model the "conditioning factors" belong to the set of the unknowns to be calculated). The *conditioning factors* that affect the conditioned category costs are *inputs* provided by the aggregate model. Therefore, for assessing the relevant disaggregate costs, it must be assumed that the aggregate route demands (or streams) have been assigned.

In calibrating disaggregate *technical* (*"free-route"*) *costs* it's also necessary to know the relevant disaggregate streams.

Summarizing, the calibration of the disaggregate technical costs requires that

(1) network constant k has been calculated;
(2) the aggregate route demands (or streams) have been assigned;
(3) the category streams (or route disaggregate demands) are also known; and
(4) in each considered link, ***one*** mean *technical* unit cost is known *for each* transportation *category* that engages any route of the link.

Also for the category technical costs the calibration may proceed link by link.

Equations from **[51]** to**[52.₃]** enable one to write:

[62.1] $\lambda_\alpha J_{\alpha i}{}^L = \mathbf{Ln}(G_\alpha{}^L / g_{\alpha i}{}^L)$, $(\forall\ i, L)$,

[62.2] $\lambda_\alpha J_{\alpha x}{}^L = \mathbf{Ln}(G_\alpha{}^L / g_{\alpha x}{}^L)$, $(\forall\ x \text{ of } L)$.

After dividing **[62.2]** by **[61.1]**, and remembering definition **[52.3]**, it is possible to isolate category technical cost $c_{\alpha x}{}^L$ as follows:

[63] $$c_{\alpha x}{}^L = \frac{(1+ k\ \mathbf{Ln} f_i{}^L)\mathbf{Ln}(G_\alpha{}^L / g_{\alpha x}{}^L)}{(1+ k\ \mathbf{Ln} f_x{}^L)\mathbf{Ln}(G_\alpha{}^L / g_{\alpha i}{}^L)}\ c_{\alpha i}{}^L .$$

In this equation, the quantities at the right hand side are known by hypothesis. Thus, all technical costs for category α , concerning routes and streams of link L, can be calculated in function of any one of them.

Equation **[63]** is obviously valid for any transportation category α and any route (or stream) x of any link L.

5.1.3 – Extended Calibration of Category Mean Costs

The surveys that measure the disaggregate streams on two routes of reference link R would be sufficient to permit also the calculation of all the network category constants λ_χ which are necessary for the use of the disaggregate model.

Under hypotheses (1), (2), (3) of Paragraph ***5.1.2,*** the determination of all category constants λ_χ gives one the possibility of calibrating the whole set of category mean technical unit costs, relevant to all the routes of the network, starting from *only one* reference category cost, say – for instance – starting from $c_{\alpha i}{}^R$ of reference link R.

In a close analogy with the procedure that leads to Equation **[63],** in fact, one can easily obtain

[64] $$c_{\beta x}{}^L = \frac{\lambda_\alpha (1+ k\ \mathbf{Ln} f_i{}^R)\mathbf{Ln}(G_\beta{}^L / g_{\beta x}{}^L)}{\lambda_\beta (1+ k\ \mathbf{Ln} f_x{}^L)\mathbf{Ln}(G_\alpha{}^R / g_{\alpha i}{}^R)}\ c_{\alpha i}{}^R ,$$

which is valid for any i of R, for any x of L, for any *link,* and for any transportation categories α e β.

6 – AUTONOMOUS USE OF THE DISAGGREGATE MODEL

In studying any transportation network, it could be required to address link disaggregate demands separately, i.e., not as implications of the aggregate assignment.

This requirement might depend on the need to plan alternative distributions of traffic categories with a view, for example, to compelling heavy load traffic to opt for certain routes instead of others, or to prevent some particular poles of the network from being used as mode exchange nodes for heavy load traffic.

Various possible examples could be given to motivate the autonomous use of the disaggregate model, *with the obvious assumption* – in such a case – *that all the transport demands on the system's links belong to the set of the model's inputs.*

Therefore, there is no logical need to connect the use of the disaggregate model to the previous use of the aggregate one. There is neither need for network constant u nor for link constants Q^L **;** whereas the network constant multiplier k is still useful and can be determined by use of second degree Equation **[18],** on the basis of the disaggregate and re-aggregate streams surveyed on the two reference route of the relevant link (this constraint does obviously persist).

The assessment of the aggregate technical costs necessary to calculate the coefficients of Equation **[18]** is naturally based on the relevant appropriate utilization of given disaggregate technical costs also in this case; but no other reference to the aggregate model is necessary. That is why the assignment of the route disaggregate demands can no more be based on equation systems **[51a], [55]** and **[57],** the parameters $G_\alpha{}^L$ being now known terms.

Actually, the assignment of the disaggregate route demand can now be carried out through iterative procedures only.

Once the network multiplier k has been calculated, the use of the *conditioning factors* $(1+ k\ \mathbf{Ln}\ d_i{}^L)$ is still possible to assess the *disaggregate conditioned costs,* but *given* aggregate route demands are not required for the conditioning factors, because the *actual* aggregate demands will result in the end of the calculations, by re-aggregation of the disaggregate demands assigned to each route through the iterative procedure.

Therefore, at the start of the iterative procedure, arbitrary values $\underline{d}_i{}^L$ may be introduced in the pertinent conditioning factors.[5]

These "arbitrary input values" change at the end of every iteration, as per route intermediate re-aggregation of the disaggregate demands assigned in first, second, third, etc. approximation, until the assigned disaggregate route demands converge on the searched values, according to the required final degree of approximation. Eventually, the per-route re-aggregation of the *autonomously* assigned *category* demands provides the (expected) aggregate route demands $d_i{}^L$, for all i and L.

Equations **[49]** to **[53]** are the main instruments of the iterative procedure**.** How these equations can work iteratively doesn't need explanation.

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[5] These initial values for the $\underline{d}_i{}^L$ are "arbitrary" from the *logical standpoint* only, which doesn't mean that the introduction of any "bizarre" values in the conditioning factors is allowed. A reasonable start for the iterative procedure could use initial $\underline{d}_i{}^L$ obtained dividing every aggregate link demand F^L (see **[57]**) by the number of the relevant routes.

Consider that all the category loads of transport on the links are now *given*: If the aggregate model were previously used to obtain assigned aggregate route demands as inputs for the *autonomous* disaggregate model, these inputs too would only work as a tentative set of "arbitrary values", since there is no logical connection between these inputs and the conditioned costs adopted for the *autonomous* disaggregate assignment. In fact, the re-aggregation of the disaggregate demands *autonomously assigned* by the iterative calculation will - route by route - give aggregate route demands noticeably different from those of the aggregate model's output.

What is clear, the autonomous use of the disaggregate model requires a remarkably higher number of input data, which normally implies a lower level of accuracy of the model's performance. It goes without saying that the greater the input data set the greater the expectable margin of error in the output.

6.1 – A Hybrid Use of the Model

A hybrid use of the model is also possible, which combines the aggregate model with the disaggregate model, leaving part of the latter autonomous.

This hybrid use may be useful when the transportation study must also focus, because of special issues pointed out by the analysis of the system, on a restricted section of the overall transportation network. For example, there might be the need to re-direct the category traffic distribution as far as two or three links of the system are concerned, whereas no particular problem has to be accounted for concerning the rest of the system.

In a case like that, *disaggregate* transport demands are required for a limited number of links only, while the transport demands for each of the remaining links may be given in an aggregate form.

Therefore, the model application may firstly proceed as if not regarded by the autonomous use of the disaggregate model.

Subsequently, the iterative procedure proper to the autonomous use of the disaggregate model may be limited to the network sections concerned, so that *exogenous data for the disaggregate transport demands are required only for the few links* regarded by particular planning problems.

It is worth remarking that the possibility of a hybrid use of the model could result as appropriate in an ample range of practical cases, leaving a remarkable degree of methodological flexibility to transportation analysts and planners.

Thus, the application of the model enables transportation decision makers to carry out a sequence of tests aimed at verifying the suitability of alternative planning hypotheses and policy objectives.

NOTE

With reference to Paragraph **4.3,** the two pages that follow display tables built in Excel software, which show how it is possible to set up algebraic matrices for the calculation of disaggregate transportation demands by use of the output from the aggregate model.

The tables are mere numerical examples that don't relate to any particular application of the model. What matters is grasping the basic criteria that drive the composition of the calculation matrices.

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ANNEX TO THE TEXT CONCERNING THE TRANSPORTATION MODEL - REFER TO PARAGRAPH 4.3

CALCULATION OF DEMAND FOR DISAGGREGATE TRANSPORT

Calculation for streams of two categories (*a* and *b*) distributed among four routes (*1, 2, 3, 4*) of link *S*

(This numerical exercise does not relate to any particular application of the model)

MATRIX OF THE EQUATION SYSTEM FOR LINK S — Known terms

	G_a^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S	d_i^S
(*)	0.0987	-1	0	0	0	0	0	0	0	0	0
	0.4320	0	-1	0	0	0	0	0	0	0	0
	0.1660	0	0	-1	0	0	0	0	0	0	0
	0.3033	0	0	0	-1	0	0	0	0	0	0
	0	0	0	0	0	0.4440	-1	0	0	0	0
	0	0	0	0	0	0.0987	0	-1	0	0	0
	0	0	0	0	0	0.3450	0	0	-1	0	0
	0	0	0	0	0	0.1123	0	0	0	-1	0
	0	1	0	0	0	0	1	0	0	0	122
	0	0	1	0	0	0	0	1	0	0	321
Σ =	1										

Determ. **-0.18207**

(*) **Note:** The summation of the coefficients of unknown variables G_i^S must equal 1.

Calculation matrix

d_i^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0	-1	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
122	1	0	0	0	0	1	0	0	0
321	0	1	0	0	0	0	1	0	0

716.676 G_a^S

G_a^S 716.68 — Determ. -130.483

Calculation matrix

G_a^S	g_{a1}^S	d_i^S	g_{a3}^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	0	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	122	0	0	0	1	0	0	0
0	0	321	0	0	0	0	1	0	0

309.6 g_{a2}^S

Determ. -56.37

Calculation matrix

G_a^S	d_i^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	0	0	0	0	0	0	0	0	0
0.432	0	-1	0	0	0	0	0	0	0
0.166	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	122	0	0	0	0	1	0	0	0
0	321	1	0	0	0	0	1	0	0

70.7369 g_{a1}^S

g_{a1}^S 70.736 — Determ. -12.8786

Calculation matrix

G_a^S	g_{a1}^S	g_{a2}^S	d_i^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	0	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	0	122	0	0	1	0	0	0
0	0	1	321	0	0	0	1	0	0

118.97 g_{a3}^S

Determ. -21.86

Calculation matrix

G_a^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	d_i^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	-1	0	0	0	0	0	0	0	0
0.432	0	-1	0	0	0	0	0	0	0
0.166	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	0	-1
0	1	0	0	0	122	1	0	0	0
0	0	1	0	0	321	0	1	0	0

115.46 G_b^S

G_b^S 115.46 — Determ. -21.0213

Calculation matrix

G_a^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	d_i^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	0	0	122	0	1	0	0	0
0	0	1	0	321	0	0	1	0	0

217.37 g_{a4}^S

Determ. -39.56

G_a^S = **716.676 = summation of the 4 *a*-category streams**

Known terms — EQUATION SYSTEM MATRIX

d_i^S	G_a^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	G_b^S	g_{b1}^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0	0.0987	-1	0	0	0	0	0	0	0	0
0	0.432	0	-1	0	0	0	0	0	0	0
0	0.166	0	0	-1	0	0	0	0	0	0
0	0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0	0.1123	0	0	0	-1
122	0	1	0	0	0	0	1	0	0	0
321	0	0	1	0	0	0	0	1	0	0

Determ. **-0.18207**

Calculation matrix

G_a^S	g_{a1}^S	g_{a2}^S	g_{a3}^S	g_{a4}^S	G_b^S	d_i^S	g_{b2}^S	g_{b3}^S	g_{b4}^S
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	0	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	0	0	0	0	122	0	0	0
0	0	1	0	0	0	321	1	0	0

61.264 g_{b1}^S

Determ. -9.333

Calculation matrix

G_a^3	g_{a1}^3	g_{a2}^3	g_{a3}^3	g_{a4}^3	G_b^3	g_{b1}^3	d_1^3	g_{b3}^3	g_{b4}^3
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	0	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	0	0	0	0	1	122	0	0
0	0	1	0	0	0	0	321	0	0
11.3888	g_{b2}^3								

Determ.	-2.0748

Calculation matrix

G_a^3	g_{a1}^3	g_{a2}^3	g_{a3}^3	g_{a4}^3	G_b^3	g_{b1}^3	g_{b2}^3	d_1^3	g_{b4}^3
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	0	0
0	0	0	0	0	0.1123	0	0	0	-1
0	1	0	0	0	0	1	0	122	0
0	0	1	0	0	0	0	1	321	0
39.834	g_{b3}^3								

Determ.	-7.252

Calculation matrix

G_a^3	g_{a1}^3	g_{a2}^3	g_{a3}^3	g_{a4}^3	G_b^3	g_{b1}^3	g_{b2}^3	g_{b3}^3	d_1^3
0.0987	-1	0	0	0	0	0	0	0	0
0.4320	0	-1	0	0	0	0	0	0	0
0.1660	0	0	-1	0	0	0	0	0	0
0.3033	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0.4440	-1	0	0	0
0	0	0	0	0	0.0987	0	-1	0	0
0	0	0	0	0	0.3450	0	0	-1	0
0	0	0	0	0	0.1123	0	0	0	0
0	1	0	0	0	0	1	0	0	122
0	0	1	0	0	0	0	1	0	321
12.9861	g_{b4}^3								

Determ.	-2.38069

G_s^3 = 115.46 = Summation of the g_{bi}^3 calculated.

Example of equation system matrix for a different link *S*, in which there are 4 load categories that engage 3 routes

	G_a^3	g_{a1}^3	g_{a2}^3	g_{a3}^3	G_b^3	g_{b1}^3	g_{b2}^3	g_{b3}^3	G_c^3	g_{c1}^3	g_{c2}^3	g_{c3}^3	G_d^3	g_{d1}^3	g_{d2}^3	g_{d3}^3
1	0.3210	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.1230	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0.5560	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0.4030	-1	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0.0987	0	-1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0.4983	0	0	-1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0.2320	-1	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0.4440	0	-1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0.3240	0	0	-1	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0.0789	-1	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0.7980	0	-1	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0.1231	0	0	-1
13	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
14	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
15	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
16	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

Known terms

d_i^3
0
0
0
0
0
0
0
0
0
0
0
0
d_1^3
d_2^3
F^3
d_3^3

Coefficients of equation 16 are a linear combination of the other equations' coefficients. Thus, this line of coefficients - along with the relevant column - must be eliminated from the system's matrix.

coeff. G_a^3
0.3210
0.1230
0.5560
Σ = **1.0000**

coeff. G_b^3
0.4030
0.0987
0.4983
Σ = **1.0000**

coeff. G_c^3
0.2320
0.4440
0.3240
Σ = **1.0000**

coeff. G_d^3
0.0789
0.7980
0.1231
Σ = **1.0000**

The two examples above show how the combination of number of transport categories and number of routes might vary from link to link of any transportation network. Therefore, the equations for the determination of disaggregate demands – when based on the output of the aggregate model ─ shall suitably be identified link by link.

The second example of the two, which considers 4 categories on 3 routes, shows that one of the equations is a linear combination of the other equations: as a consequence, the determinant of the main matrix of the equation system is nil. In certain cases this may happen for more than one equation, depending on the combination of number of transport categories and number of routes in the link. In such cases, to solve the simultaneous equation system, one or more of the unknowns must in some way be determined and transferred – after multiplication by the relative coefficient and change of the algebraic sign – to the column of the known terms, and the relevant equations must be eliminated.

These parameters to be taken as additional known terms may regard either category route demands (or streams) or category transport demands on the link. Obviously, the choice to be made depends on the particular circumstances and on the data that are available during the application of the model.

(*Subsequent paragraphs focus on some aspects of the model that are of a particular interest to regional planners*).

7 - AN APPENDIX FOR PLANNERS

Traffic generated by human activities makes sense to the extent to which it helps achieving benefits, which are pursued with a view to improving the quality of life in human communities.

As already observed in Paragraph **2.6,** it would be much more reasonable, in assessing the model inputs, to associate accurate estimates of mean unit benefits with the travels undertaken by the network's users, instead of or together with mean unit transportation costs. But this is practically impossible, especially because the expected individual benefits have a range of variability and indeterminacy which is much wider than that of the transportation costs, while there is no practical means to satisfy the purpose in a useful or significant way. That is why the model hypothesizes the existence of a "mean" benefit that motivates every travel, but the model doesn't provide any practical criterion for the correct determination of that input.

However, regional planners, who cooperate with transportation experts, need to know what - in terms of *general state* of the study system - is the meaning of any traffic assignment. If – working upon planning hypotheses – one considers any traffic assignment different from that provided by an application of the model based on the available data, how is it possible to establish whether alternative assignments can improve or worsen the existing *overall state* of the system?

Any reasonable answer to questions like that might somehow compensate the impossibility of assessing modifications in the net benefits acquired through the use of the transportation network.

Among current criteria used in transportation studies to make comparative quality evaluations of the network's functional conditions, there is the search for traffic assignments that minimise the transportation costs. Often, such a criterion determines the adoption of the so-called "*objective-function*", whose optimisation provides the analytical procedures for the assignment of traffic streams. However, such a criterion is questionable to the extent that minimisation of transportation costs does not *per se* imply maximisation of the benefits expected by the users of the transportation network. Within a more general conceptual framework, transportation costs, like any other kind of production costs, should be viewed as *investments*, and investment levels should always be related to the levels of the benefits pursued. Thus, an overall increase in the transportation costs, which may depend – for instance – on the use of more expensive infrastructure and/or transportation technology, could be associated with *much* more rewarding benefits.

One possible way for evaluating the *state* in which the transportation system works could be suggested by an appropriate systems analysis. In this connection, it's worth mentioning that the analytical methodology used to build this transportation model is taken from a theory of systems formulated by the same author [6]. According to that theory, the alterations in the relationships between the components of any complex system entails both modifications of the system's state and evolution (or involution) processes, which can improve (or worsen, respectively) the system's degree of intrinsic organisation. The theory does also show that the system's level of organisation is the response to the *intents* that motivate the system's activity. Actually, the higher the system's level of intrinsic organisation the higher the degree of satisfaction of the expectations associated with the system's activities.

The activities of a complex system may be those represented, for example, by measurable interactions between persons of a community, or between human settlements, or between production sectors of an economic system, and so on.

Traffic streams between poles of any transportation network can also be considered as interactions between components of a complex system.

In order to proceed on with the analysis of the "behaviour" of the transportation system, it's convenient to adopt some of the symbols used by the mentioned theory, which requires a *translation* of the symbols used in the transportation model into the theory's relevant symbols.

7.1 – Transportation Network as a Complex System

In the transportation model, the transport demand on any link generates a transport flow from an *origin pole* (say Pole *j*) to a *destination pole* (say Pole *k*), whatever the distribution of this flow among the link's routes.

In the model above, symbols like F^L are used to identify the transport flow on any link *L*, whereas – from now on – it is more convenient to denote the transport flow between any two poles, say from *j* to *k*, with symbol T_{jk}. For example, if link *L* is the link that connects *j* to *k*, we can now write, remembering definition **[2],**

[7.1] $$T_{jk} = F^L = \sum_{i=1}^{r} f_i^L = \sum_{i=1}^{r} f_{jk}^{\,i}, \qquad \forall\ i,j,k,r,L,$$

r being the number of routes in the link. Relation **[7.1]** establishes also the new symbolic equivalence $f_i^L = f_{jk}^{\,i}$, i.e., the new way in which any aggregate traffic stream is now denoted, since $f_{ik}^{\,i}$ is the new symbol adopted to indicate the stream that uses route *i* to go from Pole *j* to Pole *k*.

With reference to the conventional time unit, symbol O_j is used to express all the traffic flows that move from *j* toward *all* the network's poles, and symbol D_k is used to express *all* the traffic flows bound for *k* from *all* the network's poles. Therefore, the following definitions are established:

[6] *M. Ludovico,* "L'evoluzione sintropica dei sistemi urbani" *(Syntropy and Evolution of Urbam Systems)*, Bulzoni, Rome 1991, ***op.cit***. A summary of this theory is under "*A Theory of Complex Systems"* in this web-page.

[7.2] $$O_j = \Sigma_k T_{jk} = \Sigma_k \Sigma_i f_{jk}^{\,i} \qquad \forall j,$$

[7.3] $$D_k = \Sigma_j T_{jk} = \Sigma_j \Sigma_i f_{jk}^{\,i} \qquad \forall k,$$

whence, remembering definitions **[2]** and **[7.1]**, one can write:

[7.4] $$\Sigma_j O_j = \Sigma_k D_k = \Sigma_j \Sigma_k T_{jk} = F.$$

The probability that a transport unit, in the fixed time unit, has its origin in any Pole j and is bound for any other Pole k (previously expressed by definition $P^L = F^L/F$) can now be expressed by the following new definition:

[7.5] $$P_{jk} = P^L = T_{jk}/F, \qquad (j \neq k, \quad \forall j,k),$$

in which link L is the link that connects Pole j to Pole k.

It's the new way to symbolise the probability of the demand for transport on a link. It's also the set of the N^2-N probabilities of a *distribution*, so as to give

[7.6] $$\Sigma_j \Sigma_k P_{jk} = 1, \qquad (i, j = 1, 2, \ldots N^2\text{-}N\,;\; j \neq k).$$

Beside this probability distribution, there is the other probability distribution that is associated with the distribution of the traffic streams amongst all the routes of the transportation network. If we define the following probability

[7.7] $$p_{jk}^{\,i} = p_i^{\,L} = f_i^L/F = f_{jk}^{\,i}/F, \qquad (j \neq k, \quad \forall j,k),$$

it's also true that

[7.8] $$\Sigma_j \Sigma_k \Sigma_i\, p_{jk}^{\,i} = 1,$$

in which superscript i (corresponding to subscript i in $p_i^{\,L}$) indicates any route, the number of routes varying in general from link to link. Probability $p_{jk}^{\,i}$ is the probability that any traffic unit in the transportation network, and in the fixed time unit, goes route i of the link from j to k.

Therefore, two different *probabilistic uncertainties,* or *entropies,* $\in$ and $E,$ can be associated with probability distributions **[7.5]** and **[7.7]**, respectively. Namely:

[7.9] $$\in = -\,\Sigma_j \Sigma_k\, (P_{jk}\,\mathbf{Ln}\,P_{jk}),$$
and
[7.10] $$E = -\,\Sigma_j \Sigma_k \Sigma_i\, (p_{jk}^{\,i}\,\mathbf{Ln}\,p_{jk}^{\,i}).$$

It's easy to prove that it's always $\in \leq E,$ because

[7.11] $$P_{jk} = \Sigma_i\, p_{jk}^{\,i}, \qquad (j \neq k, \quad \forall j,k),$$

and – given the distribution of transports on links – the stream distribution amongst the network's routes increases the number of elements of the probability distribution while respecting constraint **[7.11]**. In general and under given constraints, the entropy associated with any probability distribution increases with the number of the distribution's elements.

The theory proves that the system's theoretical maximum entropy is associated with a distribution of probabilities in which all the probabilities are equal to each other. Considering our transportation network, the system's maximum entropy would be expressed by a uniform distribution of streams amongst all the network's routes, so as to verify the condition expressed by the following relation:

[7.12] $$p_{jk}^{\,i} = p_{qs}^{\,n} = p = 1/R = \text{constant},$$

relevant to any $j, k, q, s, i, n,$ in which R is the total number of routes in the network. In such a case, the system's maximum entropy is expressed by

[7.13] $$E_{max} = -\sum_{1}^{R} (p\,\mathbf{Ln}\,p) = -R(1/R)\mathbf{Ln}(1/R).$$

It's another form of Equation **[12]**, Paragraph **2.7,** since

[7.13a] $$E_{max} = \mathbf{Ln}\,R = \mathbf{Ln}\,\Pi.$$

According to the theory of complex systems recalled above, "maximum entropy" means maximum degree of randomness or disorder in the system. Thus, a certain degree of organisation, or level of ***syntropy*** in the system, can be associated with any probability distribution which differs from distribution **[7.12].** Therefore, the system's degree of intrinsic organisation, or *syntropy*, can be expressed as the difference between its possible maximum entropy E_{max} and its *actual* entropy level $E,$ i.e.,

[7.14] $$S = E_{max} - E.$$

Furthermore, still referring to the recalled systems theory, there is an interesting *anomalous* distribution of transport flow probabilities $P_{jk}^{\,*}$ (or, correspondingly, of transport demands on link), which is obtained when the probability distribution is solely subject to constraint Equations **[7.2], [7.3]** and **[7.6]**. In other words, if there is neither cost nor benefit constraint to condition the users' options, then the choice of every travel destination depends only on the amount of travels O_j that start from Pole j (any j), and on the amount of travel arrivals D_k expected in Pole k (any k). In a situation like that, the probability of transport from any j to any k is expressed by

[7.15] $$P_{jk}^{\,*} = \frac{T_{jk}^{\,*}}{F} = \frac{O_j\,D_k}{F^2}, \qquad (\forall j,k)$$

while

[7.15a] $$p_{jk}^{\,*} = P_{jk}^{\,*}/r_{jk}$$

expresses the relevant uniform distribution of stream probability among the routes of the link; r_{jk} being the number of routes inherent in the link.

The probabilistic uncertainty, or *entropy*, associated with this quite special probability distribution is then given by

[7.16] $$E^* = -\,\Sigma_j \Sigma_k\, (P_{jk}^{\,*}\,\mathbf{Ln}\,p_{jk}^{\,*}).$$

(It's worth drawing attention to the two different symbols $P_{jk}^{\,*}$ and $p_{jk}^{\,*}$ used in **[7.16],** which must be considered according to definitions **[7.15]** and **[7.15a],** respectively).

The theory shows that this *anomalous* distribution of flow probability (which is never observed as an actual one) plays a significant role in the systems analysis, since it is a necessary ingredient for measuring the *level of stability* of the system. *Anomalous* probability distribution **[7.15]** does obviously imply, in correspondence with entropy E^*, a particular level of syntropy S^*, which is expressed by

[7.17] $$S^* = E_{max} - E^*.$$

This is the specific level of the system's organisation that pertains *only* to the territorial concentrations and locations of the transport origins (O_j) and destinations (D_k).

E^* and S^* are refereed to as *base entropy* and *base syntropy* of the system, respectively.

The theory proves that *base syntropy* S^* gives the measurement of the system's *stability*, i.e., the system's *degree of reluctance to change* the state that defines the hierarchy (i.e. the respective importance) of the interacting poles.

Whatever the configuration of the traffic streams between the poles of the system, the system doesn't modify its intrinsic *stability* if the territorial location and local density distribution of the *travel sources* and *destinations* keeps unchanged.

Instead, not only does the system's stability change, but also an evolution or involution process in the system is triggered by any minor, though *irreversible*, change in the territorial distribution of the O_js and D_ks. The logical demonstration of this is in the cited text of the theory.

"Evolution" and "involution", in this context, do mean an increase and a decrease, respectively, in the system's level of intrinsic organisation (or *syntropy*).

It's worth observing that **[7.14]** and **[7.17]** imply also

[7.18] $$S - S^* = E^* - E,$$

as well as

[7.19] $$S + E = S^* + E^* = E_{max} = H = \text{constant},$$

which means that any increase in the system's level of intrinsic organisation implies an equivalent decrease in the system's entropy/disorder.

Constant H is a characteristic of the study system, and makes the system work in a way similar to that proper to any field of conservative energy.

Constant H is referred to as *the system's potential*, since the system's absolute maximum degree of organisation and the absolute maximum degree of disorder have an identical numerical value. If $E = 0$, then $S = H$. Symmetrically, if $E = H$, then $S = 0$. However, as the theory proves, H is only a *limit-value,* for both syntropy and entropy can only achieve that value asymptotically, under quite unlikely conditions.

7.2 – Travel Benefits in Terms of System Syntropy

The constraint equations that have been written to formulate the transportation model (refer to Paragraphs **2.6** and **3.1**) contain the amount of information that determines the actual level of the system's syntropy. The *intent* that motivates any transport unit is a direct linear function of the mean net benefit $u_{jk} = B^L - J^L$ (or $u_{jk} = B_{jk} - J_{jk}$) expected by any user of link j_k of the transportation network (refer to Paragraph **2.6**). The theory synthesizes this important aspect of the model by proving that an average measurement of this motive *intent* is expressed, concerning any link j_k, by

[7.20] $$\mu_{jk} = \lambda u_{jk} = S^* + \mathbf{Ln}(T_{jk}/T_{jk}^*), \qquad (\forall\, j,k)$$

in which λ is a network constant that depends on the measurement system adopted, $T_{jk}^* = O_j D_k / F$ (as per definition **[7.15]** above), and T_{jk} is the transport on the link that joins j to k.[7]

In a close analogy with equation **[7.20],** the mean *intent* associated with every stream unit on the link's routes is expressed by

[7.21] $$\mu_{jk}^{\,i} = \lambda u_{jk}^{\,i} = S^* + \mathbf{Ln}(f_{jk}^{\,i}/f_{jk}^{\,*}), \qquad (\forall i)$$

in which $f_{jk}^* = T_{jk}^* / r_{jk}$,

r_{jk} still being the number of routes inherent in the link.

Quantities μ_{jk} and μ_{jk}^i, which are labelled "intents", represent *net benefits,* each expressing a quantity proportional to the difference between a *gross unit benefit* and a *unit cost.* As clearly visible in the equations that define them, these quantities have no physical dimension, as they depend exclusively on the system's configuration. The system's configuration is described by the distribution of interaction probability, whatever the measurement criteria adopted for quantifying the interactions (i.e., the traffic streams). Therefore, all *the quantities in Equation* **[21]** are pure numbers and *express absolute values,* which are provided by ratios between homogeneous physical quantities.

Equations **[7.20]** and **[7.21]** are an indication of the importance of the *anomalous* probability distributions p_{jk}^* in quantifying the "benefit functions" μ_{jk} and μ_{jk}^i. Consider that the ratios T_{jk}/T_{jk}^* and f_{jk}^i/f_{jk}^* are identical to the ratios between the respective probabilities. There is to note, in particular, the contribution of stability S^* (or *base syntropy*) to the determination of the benefit functions that motivate each travel.

Equation **[7.21]** gives evidence to a major feature of the theory. Currently, various monetary criteria are in use to quantify economic benefits that derive from any kind of human activity. At variance with that, a basic achievement of the theory consists of returning to the determination of *absolute* benefits through the measurement of the activities themselves; and, concerning transport systems in particular, to the measurement of the comparative intensity and distribution of the *interactions* (traffic streams) between the system's principal components.

[7] Constant λ in this equation has no relation with the category constants λ_α that have been introduced in the transportation disaggregate model.

In practical applications of the theory, there is no actual need for the numerical determination of this λ, since it is convenient to measure benefits in terms of *intents*, as per the relative definitions given by **[7.20]** and **[7.21]**.

In assessing the state of the interaction system in terms of mean general ***absolute*** *net benefit* (*absolute utility,* or *system intent*) enjoyed by the system's users/activators, the theory proves and provides the following simple formula

[7.22] $$\mu = \lambda u = \lambda U / F = S ,$$

(in which

$$U = \sum_j \sum_k \sum_i (u_{jk}^{\,i} f_{jk}^{\,i}) = SF / \lambda$$

does express the overall *net* ***economic*** *benefit* [8] that can be associated with the system's activity, and S is the system's *syntropy*, as defined by Equation **[7.14]** through Equations **[7.10]** and **[7.13a]:** As expected, under the same level of *syntropy*, this overall economic benefit is directly proportional to the volume F of the system's internal interactions**).**

Thus, a direct and immediate equivalence is established by Equation **[7.22]** between the mean *absolute* net benefit that the system's activity generates and the system's level of intrinsic organisation (*syntropy*) S . [9]

Therefore, a useful instrument is so available to make significant comparisons between different analytical assignments of traffic streams among the routes of any transportation system.

7.3 – Conclusions

In the light of the preceding notes, the use of the transportation model becomes particularly useful in coping with regional planning issues.

The functional role of "poles" and "links" in a transporttation network can be planned by targeted adjustment designs of the infrastructure as well as of project implementation timing.

The theoretical arguments expounded in this essay provide the analytical support that allows planners to predict, on a logical basis, what *better or worse effect* – in terms of overall benefits to the users – one can reasonably expect from hypothesised or requested changes in the technical and functional characteristics of the study transportation system.

In this connection, it's interesting to observe that technical and functional characteristics of any kind of transportation infrastructure and mode are eventually and substantially expressed in terms of mean transportation costs involved by the relevant use. And the assessment of these costs, whatever the criterion adopted, is practically all the work that's needed to apply the model and to achieve meaningful assessments of the benefits generated by the transportation activities.

Further important developments of the concepts introduced here, especially concerning evolution processes subsequent to changes in the system's configuration, are discussed by the recalled systems theory, to which the reader is invited to appeal for a more comprehensive and effective utilisation of this model in planning activities.

[8] **"Economic** benefit" is the benefit expressed in terms of economic goods or in monetary terms, at variance with the "**absolute** benefit", which is instead expressed by a pure number.

[9] It's interesting to comment on the meaning of coefficient λ as this results from Equation **[7.22]:** $\lambda = S / u$ indicates the amount of overall organisation in the system that is necessary to achieve a given level u of mean net *economic* benefit. Thus, λ seems to be the degree of intrinsic "viscosity" or "inefficiency" of the study system. Conversely, under the same level of intrinsic organisation S, the mean net economic benefit produced by the system is *inversely* proportional to the degree λ of intrinsic "viscosity" or "inefficiency" of the system.

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